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DEVELOPMENT OF METHODS OF ANALYTICAL GEOMETRY OF A SPHERE FOR SOLVING GEODESY AND NAVIGATION TASKS

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The article develops ideas and formulas of analytical geometry for spherical surface of the Earth globe in relation to main tasks of global geodesy and navigation. It examines peculiarities of sphere inner geometry and properties of its primary, secondary and higher-order curves. It was proved that spherical hyperbola and parabola are spherical ellipses with specific parameters. The Cartesian ordinates were introduced into the sphere and the relation between them and polar spherical coordinates was established. With the help of central projection of sphere points on tangential plane the corresponding elliptical plane with beltrami ordinates was introduced. The article describes main formulas of analytical geometry for projected elliptical plane, which correspond to geometry of projected sphere. It also introduces several formulas for primary, secondary and higher-order curves for this sphere.

Key words**:** sphere, analytical geometry, inner geometry, elliptical plane, Riemann geometry, Cartesian ordinates, polar coordinates, tangential coordinates

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Introduction. Geodesy, cartography, radio navigation, radar-location, ship navigation and aviation usually solve geometrical tasks for Earth surface using a certain reference ellipsoid with geodesic coordinates (B, L, H). The main method for solving these tasks is making relevant constructions on equivalent sphere with following corrections due to differences between reference ellipsoid and sphere [4, 15]. In turn the geometrical tasks for sphere are solved using spherical trigonometry [1, 2, 8, 16]. These tasks usually use transcendental equations and do not have quadrature solutions.

For Euclidian plane with Cartesian ordinates there exists a very efficient mathematical tool, i.e. analytical geometry, which enables to reduce plane geometry tasks to mathematical analysis tasks and simplify solutions. These coordinates could be used for a sphere and different geometrical tasks would be reduced to mathematical analysis, – beltrami coordinates [6, 7, 17], which significantly resolves problem with transcendental equations.

The fundamental mathematics uses these constructions to reduce spherical analytical geometry to Riemann geometry for elliptical plane, i.e. on two-dimensional variety of constant positive curvature. However, the tools of non-Euclidian geometry for elliptical plane is too abstract and rarely used for technical issues.

The aim of this article is to describe main ideas and formulas of analytical geometry for sphere in order to use them for solving practical tasks with the help of elementary constructions available to engineers with usual skills in mathematics.

Peculiarities of sphere inner geometry. As it is known, any smooth surface inner geometry is based on geometrical constructions, which could be created using geometrical elements of this surface [5, 7, 17].

The initial concept of inner geometry is length, and other constructions of inner geometry using the ability to consider small shapes of the surface at Euclidian plane, which is tangential to this surface. For example, the angle between two curves of the surface is the angle between their projections on tangential plane, which is constructed at the intersection of these curves.

In case of spherical surface the Euclidian curves are geodesic lines, which are the result of intersection of sphere and plane going through sphere center C of R_0 radius. These lines are called great circles or spherical lines.

If all sphere points are projected from C center to the plane tangential to the southern pole of P' sphere and distances between projected points are set to the spherical distances between corresponding points of the sphere, then the resulting plane W will be a two-dimension Riemann variety of constant positive curvature, and inner geometry of this plane is a Riemann geometry for elliptical plane [6, 17].

All great circles of a sphere are projected on to elliptical plane as lines. The main difference of sphere inner geometry from elliptical plane is that antipodal points correspond to one point of elliptical plane.

Let us consider the main features of sphere inner geometry [3, 5]:

1) two lines cross antipodal points form two digons with apex angles γ and $\gamma' = 2\pi - \gamma$;

2) a point spaced apart from all points of this line at spherical distance $l_0 = \pi R_0/2$ is called a pole P of this line; there are two such points and they are antipodal;

3) the angle between two spherical lines is measured by distance between their poles divided by sphere radius;

4) geometrical locus of points located at the spherical distance $l = \pi R_0/2$ from this point is a spherical line that is called a pole of this point; any line drawn through this point is perpendicular to this pole point: this line is called normal for this line;

5) any normal lines of any line intersect in two points: poles of this line;

6) any two lines of a sphere have base normal, which pole is an intersection point of these lines.

If we draw geodesic spherical lines through three points of a sphere, which are not pair-wise antipodal, then we have a spherical triangle. We can build eight spherical triangles using these three points. In Riemann geometry (at elliptical plane) there are only four such triangles because antipodal points of the matching spheres are considered being identical.

If length of triangle sides is smaller than half-length of a great circle and apex angles are less than π, then this triangle is called Euler.

Since there is no concept of parallelism in sphere inner geometry, it does not have the concept of uniformity. That is why two Euler triangles with three pair-wise identical angles are equal or mirror symmetric.

The sum of inner angles of Euler triangle is always bigger than π, but smaller than 3π. For a triangle ABC with angles A, B and C the spherical excess is $\varepsilon = A + B + C - \pi$.

The cosine theorem for spherical triangles is formulated in the following way: $\cos(AB/R_0) = \cos(BC/R_0)\cos(AC/R_0) + \sin(BC/R_0)\sin(AC/R_0)\cos C$.

In right spherical triangle with sides AC, BC and hypotenuse AB there are correct equations [2, 3]:

$$\cos(AB/R_0) = \cos(BC/R_0)\cos(AC/R_0) \text{ and } \operatorname{tg}(AC/R_0) = \operatorname{tg}(AB/R_0)\cos\alpha, \tag{1}$$

where α – angle between side AC and hypotenuse AB.

The area of a spherical triangle S_Δ is described through its side lengths a, b, c in accordance with L'Huilier theorem:

$$\operatorname{tg}\frac{S_\Delta}{4R_0^2} = \sqrt{\operatorname{tg}\frac{p}{2R_0}\operatorname{tg}\frac{p-a}{2R_0}\operatorname{tg}\frac{p-b}{2R_0}\operatorname{tg}\frac{p-c}{2R_0}},$$

where $p = (a + b + c)/2$ – semi-perimeter of Euler triangle ABC.

With $R_0 \to \infty$ we have Herone's formula for plane triangle

$$S_\Delta = \sqrt{p(p-a)(p-b)(p-c)}.$$

Since two spherical triangles with three equal angles are equidimensional we can apply Gerard formula, which does not have its analogue in plane geometry: $S_\Delta = \varepsilon R_0^2$.

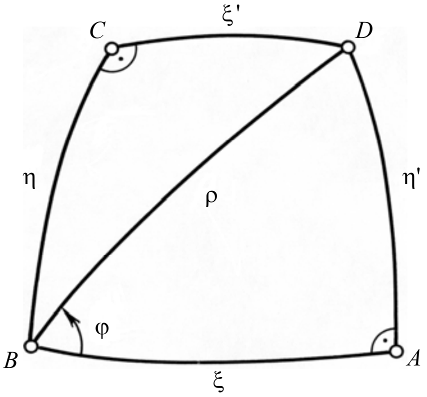

Fig.1. Spherical square with three right angles: *ABC* = *BAD* = *BCD* = 90°

Theorem 1 [13]. Assume a spherical triangle *ABCD* with right angles *ABC*, *BAD* и *BCD* (Fig.1), then we have the following relations:

$$\mathrm{tg}^2\rho = \mathrm{tg}^2\xi + \mathrm{tg}^2\eta;\ \mathrm{tg}\,\varphi = \mathrm{tg}\,\eta\,/\,\mathrm{tg}\,\xi; \tag{2}$$

$$\mathrm{tg}\,\xi' = \mathrm{tg}\,\xi\,\cos\eta;\ \mathrm{tg}\,\eta' = \mathrm{tg}\,\eta\,\cos\xi, \tag{3}$$

where $\rho = BD/R_0$; $\xi = BA/R_0$; $\eta = BC/R_0$; $\xi' = CD/R_0$; $\eta' = AD/R_0$.

Proof. From right triangles *ABD* and *CBD* using formulas (1) we have: $\mathrm{tg}\xi = \mathrm{tg}\rho\cos\varphi$, $\mathrm{tg}\eta = \mathrm{tg}\rho\cos(\pi/2 - \varphi) = \mathrm{tg}\rho\sin\varphi$ and following formulas (2). Since $\cos\rho = \cos\xi'\cos\eta$, or

$$\frac{1}{1+\mathrm{tg}^2\rho} = \frac{\cos^2\eta}{1+\mathrm{tg}^2\xi'},$$

taking into account (2), we have

$$1 + \mathrm{tg}^2\xi' = \cos^2\eta + \mathrm{tg}^2\eta\cos^2\eta + \mathrm{tg}^2\xi\cos^2\eta;\ \ \mathrm{tg}^2\xi' = \cos^2\eta + \sin^2\eta - 1 + \mathrm{tg}^2\xi\cos^2\eta,$$

i.e. relations are correct (3).

When $R_0 \to \infty$ the square *ABCD* becomes a plane square with $\xi' = \xi$, $\eta' = \eta$, $\rho^2 = \xi^2 + \eta^2$, $\mathrm{tg}\varphi = \eta/\xi = \eta'/\xi'$, i.e. the Pythagorean propositions are correct.

Set of points distanced from arbitrary point *O* on spherical distance $\breve{r} < \pi R_0/2$, form a spherical circle; point *O* is called the center of this circle, and $\breve{r}$ is its radius. Spherical ellipse is a set of sphere points, the sum of their spherical distances to two given points is called ellipse focus, which is constant and equals the length of major axis of the ellipse. The spherical hyperbola is a set of sphere points the range difference to two given points called hyperbola focuses is constant.

Theorem 2 [13]. Let us have spherical hyperbola with focuses F_1 and F_2 (Fig.2). Then this hyperbola is also an ellipse with focuses at point F_1 and antipodal point F_2 at F_2'.

Proof. In accordance with hyperbola definition $F_2M - F_1M = 2\zeta$ for any point *M* of this hyperbola, where ζ – hyperbola parameter. However, $F_2M = \pi R_0 - F_2'M$. Then, $\pi R_0 - F_1M - F_2'M = 2\zeta$ or $F_1M + F_2'M = 2(\pi R_0/2 - \zeta)$, q.e.d.

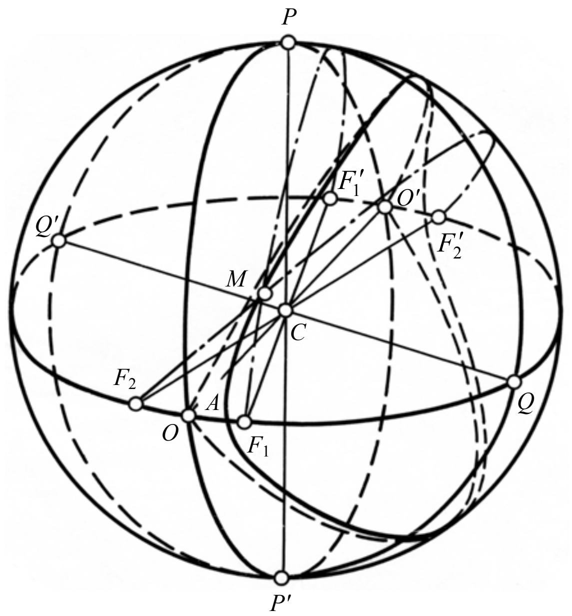

Fig.2. Spherical hyperbola

Thus, we imply that a semi-axis of the discovered ellipse: $\alpha = \pi R_0/2 - \zeta$.

If we draw a plane normal to segment *OC* (Fig.2), the resulting great circle *QPQ'P'Q* plays a role of infinity, in which spherical hyperbola touches its asymptote.

We will call a spherical parabola a set of points *M* of sphere *C*, spherical distances from them to a given point F_1 (parabola focus) and a given line (directrix *P'OP*) are equal to each other (Fig.2).

Theorem 3 [13]. A parabola of a sphere is ellipse, one of its focuses is a focus of parabola F_1, and another one is a pole of a directrix (point *Q*).

Proof. Taking into account parabola definition, we have $F_1M = KM$, where *K* is a point of intersection of line *QM* and directrix *P'OP*.

For any point *M* of this parabola the correct relation is $KM/R_0 = \pi/2 - MQ/R_0$. Thus, $F_1M/R_0 =$

$= \pi/2 - MQ/R_0$ or $(F_1M + MQ)/R_0 = \pi/2$, i.e. parabola of a sphere is an ellipse with a length of major semi-axis of $\pi R_0/4$.

Note that spherical hyperbola could always be considered as corresponding ellipse, since a digon (a distance between focuses is πR_0), a circle (a distance between focuses is zero) and a parabola are subcases of ellipses. This supports F.Klein words that in geometry of ellipse there are only two types of secondary curves: ellipse and circle [7]. Though, the last one is an ellipse subcase.

Since a sphere is a closed surface and do not have points infinitely distanced from each other many higher-order curves having infinite branches are limited: spherical cissoid, strophoid, spiral, etc.

For example, a logarithmic spiral of the sphere is a curve described using equation $\rho = a\exp(k\varphi)$, $-\infty < \varphi < \infty$. When $\varphi = 0$, $\rho = a$. When $k > 0$ and angle ranges form $\varphi = 0$ to $\varphi = \infty$ the logarithmic spiral crosses the pole of point O (when $\varphi_\infty = k^{-1}\ln[\pi R_0/(2a)]$), then reaches point O' – antipod of point O [when $\varphi' = k^{-1}\ln(\pi R_0/a)$] and continues to unwind and curl up periodically to infinity. When angle changes from $\varphi = 0$ to $\varphi = -\infty$ this spiral, just as at Euclidian plane, infinitely curls up around point O.

Sphere coordinate systems and relationship between them. There are several coordination systems for identifying the position of point M at sphere surface. More common is a system of special spherical coordinates $(\bar{r}, \bar{\varphi}, \bar{\theta})$ – with condition $\bar{r} = R_0$ and Cartesian geocentric ordinates (X, Y, Z) – with condition $X^2 + Y^2 + Z^2 = R_0^2$. Herewith axis OZ is directed from center of a sphere C to a pole P, axis OX – from point C to point O, which is an intersection of central meridian with equator, axis OY – from point C along normal to central meridian plane. These coordinates are interconnected with known equations [5, 12, 15].

However, for a unit sphere with radius $R_0 = 1$, more natural in some sense, i.e. located at the sphere surface – inner, are spherical Cartesian (ξ, η) and spherical polar (ρ, φ) coordinates as well as geographical (λ', φ'), where λ' – longitude; φ' – latitude of point M.

The position of point M in spherical Cartesian ordinates (ξ, η) is set as in Euclidian plane with distances from zero of coordinate system (point O) to an intersection of normal lines K and L, drawn from point M to mutually normal coordinate axes OQ and OP (Fig.3). The resulting spherical square $OLMK$ has three right angles: LOK, OLM и OKM.

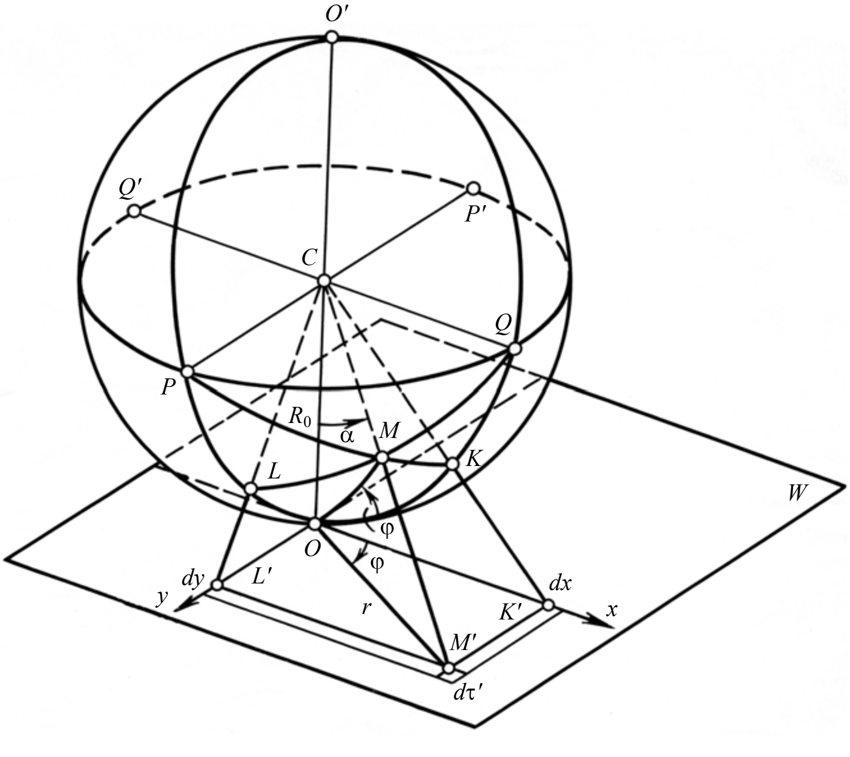

Fig.3. For establishment of definition of measure at elliptical plane

In spherical polar coordinate system (ρ, φ) the position of point M is set as length ρ of geodesic line OM and angle φ, formed by spherical polar vector OM and spherical equator $OQO'Q'O$. As it was proved before – theorem 1, formulas (2): $\mathrm{tg}^2\xi + \mathrm{tg}^2\eta = \mathrm{tg}^2\rho$; $\mathrm{tg}\varphi = \mathrm{tg}\eta / \mathrm{tg}\xi$. When $R_0 \to \infty$ the coordinate systems (ξ, η) and (ρ, φ) naturally become Cartesian (x, y) and polar (ρ, φ) at Euclidian plane.

A variety of spherical Cartesian ordinates (ξ, η) for a unit sphere is tangential coordinates (x, y), where the position of point M is set as tangents of its Cartesian ordinates: $x = \mathrm{tg}\xi$, $y = \mathrm{tg}\eta$. In tangential coordinate system, the formulas (3) are written as

$$x^2 + y^2 = \mathrm{tg}^2\rho = r^2;\ \ \mathrm{tg}\,\varphi = y/x;\ \ x = r\cos\varphi;\ \ y = r\sin\varphi, \tag{4}$$

where $r \equiv \mathrm{tg}\,\rho$.

Tangential coordinates are a special case of beltrami coordinates [6, 7, 17]. Beltrami coordinates $(x_Б, y_Б)$ are rectangular coordinates of plane W, tangential line to sphere radius R_0, which are created by projecting spherical Cartesian coordinates (ξ, η) of sphere points from its center C (Fig.3). As a result of this projection all geidesic lines of a sphere become lines of tangential plane, the geometry of this plane is non-Eucledian: Riemann geometry for elleptical plane.

Spherical Cartesian (ξ, η), beltrami $(x_Б, y_Б)$ and tangential coordinates (x, y) of a sphere of arbitrary radius R_0 are connected with the following relations:

$$x_Б = R_0\,\mathrm{tg}\,(\xi/R_0) = R_0 x,\quad y_Б = R_0\,\mathrm{tg}\,(\eta/R_0) = R_0 y;$$

$$\xi = R_0\,\mathrm{arctg}\,(x_Б/R_0) = R_0\,\mathrm{arctg}\,x,\quad \eta = R_0\,\mathrm{arctg}\,(y_Б/R_0) = R_0\,\mathrm{arctg}\,y.$$

For a unit sphere ($R_0 = 1$) tangential (x, y) and beltrami $(x_Б, y_Б)$ coordinates match.

Let us find the relation between tangential (x, y) and geographical (λ', φ') coordinates of point M at a unit sphere. From fig.1 it follows: $\lambda' = \xi$; $\mathrm{tg}\,\lambda' = \mathrm{tg}\,\xi = x$; $\varphi' = \eta'$. Thus, with equations (2) and (3), we have

$$\mathrm{tg}\,\varphi' = \mathrm{tg}\,\eta' = \mathrm{tg}\,\eta\,\cos\xi = \mathrm{tg}\eta\,/\sqrt{1+\mathrm{tg}^2\xi}\ ;$$

$$y = \mathrm{tg}\varphi'\sqrt{1+\mathrm{tg}^2\lambda'}\ ,\ \ x = \mathrm{tg}\,\lambda';$$

$$\lambda' = \mathrm{arctg}\,x;\ \ \varphi' = \mathrm{arctg}(y/\sqrt{1+x^2}\,).$$

Let us turn the coordinate system (ξ, η) counter-clockwise around point O at angle θ. For new polar coordinates $(\tilde\rho, \tilde\varphi)$ we have new obvious relations: $\tilde\rho = \rho$, $\tilde\varphi = \varphi - \theta$. Therefore, $\tilde x = \tilde\rho\cos\tilde\varphi =$ $= \rho\cos(\varphi - \theta) = x\cos\theta + y\sin\theta$.

By doing the same transformations for $\tilde y$, we find final relations between new $(\tilde x, \tilde y)$ and old (x, y) tangential coordinates of point M at the sphere, which are similar to formulas of analytical geometry for Euclidian plane:

$$\tilde x = x\cos\theta + y\sin\theta,\ \ \tilde y = -x\sin\theta + y\cos\theta;$$

$$x = \tilde x\cos\theta - \tilde y\sin\theta,\ y = \tilde x\sin\theta + \tilde y\cos\theta.$$

Due to absence of parallelism in a sphere the parallel translation of tangential coordinates has no sense. That is why in general case the translation of sphere coordinates has to be defined by a certain movement of coordinates of a new center and direction of one of the axes of the new coordinate system.

Let us turn the coordinate system (ξ, η) counter-clockwise around pole P at some angle ξ_0. Notice that value $\eta' = \varphi'$ in case of such turning stays constant. Therefore, for new coordinate systems $(\tilde\xi, \tilde\eta)$ and $(\tilde x = \mathrm{tg}\tilde\xi, \tilde y = \mathrm{tg}\tilde\eta)$ we have $\tilde\xi = \xi - \xi_0$, $\tilde\eta' = \eta'$ and $\tilde x = \mathrm{tg}\tilde\xi =$ $= (x - x_0)/(1 + x_0 x)$, $\tilde y^2/(1 + \tilde x^2) = y^2/(1 + x^2)$, where $x_0 = \mathrm{tg}\,\xi_0$. Hereafter, after identity transformations we finally have

$$\tilde x = (x - x_0)/(1 + x_0 x),\ \ \tilde y = y\sqrt{1 + x_0^2}\,\Big/(1 + x_0 x)\,; \tag{5}$$

$$x = (\tilde x + x_0)/(1 - x_0\tilde x\,);\ \ y = \tilde y\sqrt{1 + x_0^2}\,\Big/(1 - x_0\tilde x)\,.$$

The simple transformation of moving the basis of coordinate system (x, y) at point $M_0(x_0, y_0)$ and similar to parallel transformation to the plane is double turning around sphere points P and Q (Fig.3). This type of transformation can be done using two methods and resulting coordinate systems will not match.

Let us turn the coordinate system (x, y) around pole P at angle $\xi_0 = \mathrm{arctg}\,x_0$. Then new coordinates $(\tilde x, \tilde y)$ of arbitrary point M according to (5) are

$$\tilde{x} = (x - x_0)/(1 + x_0 x); \quad \tilde{y} = y\sqrt{1 + x_0^2}\Big/(1 + x_0 x).$$

The new ordinate of point M_0 is

$$\tilde{y}_0 = y_0\sqrt{1 + x_0^2}\Big/(1 + x_0^2) = y_0\Big/\sqrt{1 + x_0^2}.$$

Now let us turn the new coordinate system $(\tilde{x}, \tilde{y})$ at angle $\tilde{\eta}_0 = \operatorname{arctg}\tilde{y}_0$:

$$\bar{x} = \tilde{x}\frac{\sqrt{1 + \tilde{y}_0{}^2}}{1 + \tilde{y}_0\tilde{y}} = \frac{(x - x_0)\sqrt{1 + x_0^2 + y_0^2}}{(1 + x_0 x + y_0 y)\sqrt{1 + x_0^2}}; \quad \bar{y} = \frac{y - y_0 + x_0(yx_0 - xy_0)}{(1 + x_0 x + y_0 y)\sqrt{1 + x_0^2}}. \tag{6}$$

If we turn it first at angle η_0, and then at angle $\tilde{\xi}_0$, then we have the following expressions:

$$\hat{x} = \frac{x - x_0 + y_0(xy_0 - yx_0)}{(1 + x_0 x + y_0 y)\sqrt{1 + y_0^2}}; \quad \hat{y} = \frac{(y - y_0)\sqrt{1 + x_0^2 + y_0^2}}{(1 + x_0 x + y_0 y)\sqrt{1 + y_0^2}}.$$

Polar angles $\bar{\varphi}$ and $\hat{\varphi}$ of a certain point $M(x, y)$ in new coordinate systems $(\bar{x}, \bar{y})$ and $(\hat{x}, \hat{y})$ are defined by expressions: $\operatorname{tg}\bar{\varphi} = \bar{y}/\bar{x}$ и $\operatorname{tg}\hat{\varphi} = \hat{y}/\hat{x}$, i.e.

$$\operatorname{tg}\bar{\varphi} = \frac{y - y_0 + x_0(yx_0 - xy_0)}{(x - x_0)\sqrt{1 + x_0^2 + y_0^2}}; \quad \operatorname{tg}\hat{\varphi} = \frac{(y - y_0)\sqrt{1 + x_0^2 + y_0^2}}{x - x_0 - y_0(x_0 y + y_0 x)}.$$

Then their difference $\gamma = \bar{\varphi} - \hat{\varphi}$ is defined by formula

$$\operatorname{tg}\gamma = \operatorname{tg}(\bar{\varphi} - \hat{\varphi}) = \frac{\operatorname{tg}\bar{\varphi} - \operatorname{tg}\hat{\varphi}}{1 + \operatorname{tg}\bar{\varphi}\operatorname{tg}\hat{\varphi}}.$$

By substituting the abovementioned expressions for $\operatorname{tg}\bar{\varphi}$ and $\operatorname{tg}\hat{\varphi}$, after identity transformations we have

$$\operatorname{tg}\gamma = -\frac{x_0 y_0}{\sqrt{1 + x_0^2 + y_0^2}}.$$

When $R_0 \to \infty$ variable $\gamma \to 0$. Therefore, within this limit (at Euclidian plane) the parallel translation of Cartesian coordinates (x, y) in both cases led to one and the same coordinate system: $\bar{x} = \hat{x} = x - x_0$, $\bar{y} = \hat{y} = y - y_0$.

Definition of measure at a sphere and elliptical plane. Let us return to Fig.3. and build at plane W a rectangular coordinate system (x, y). Its origin (x, y) matches the mutual point O of sphere C and plane W. The axis Ox is directed along the intersection line of plane W and plane of spherical line OQO', and axis Oy is directed along intersection line of spherical line OPO' and plane W. The coordinates Ox and Oy of point $M'(x, y)$ at plane W will be made nonuniform, as in Cartesian ordinates for Euclidian plane, and defined by lengths of OK' and OL' of central projections of points K and L of spherical Cartesian ordinates (ξ, η) of arbitrary sphere point M. Then

$$x = OK' = R_0\operatorname{tg}(OK/R_0) = R_0\operatorname{tg}(\xi/R_0);$$

$$y = OL' = R_0\operatorname{tg}(OL/R_0) = R_0\operatorname{tg}(\eta/R_0).$$

In this case, spherical Cartesian coordinates (ξ, η) of sphere arbitrary point M, defined by lengths of segments OK and OL, and rectangular coordinates (x, y) of projection M' of point M at plane W will be correlated to each other.

Actually, we have built a system of beltrami coordinates at elliptical Riemann plane W. The variable R_0 is a curve radius of this plane. Since when $R_0 = 1$ beltrami and tangential coordinates match, the built coordinate system (x, y) will be called tangential.

Notice that angles KOM and $K'OM'$ are equal to each other (Fig.3). Therefore it is useful to introduce one more coordinate system: polar-tangential (r, φ), where $\varphi = K'OM'$, $r = OM' = R_0 \,\mathrm{tg}\,(OM/R_0) = R_0 \,\mathrm{tg}\,\alpha = R_0 \,\mathrm{tg}\,(\rho/R_0)$. According to equalities (4): $r^2 = x^2 + y^2$, $\varphi = \mathrm{arctg}\,(y/x)$; $x = r \cos\varphi$, $y = r \sin\varphi$. When $R_0 \to \infty$ system (r, φ) transforms into usual polar coordinate system at Euclidian plane (ρ, φ).

Let us define the distance l between projections $M_1'(x_1, y_1)$ and $M_2'(x_2, y_2)$ of two sphere points $M_1(\xi_1, \eta_1)$ and $M_2(\xi_2, \eta_2)$ in tangential coordinate system (x, y) at plane W. In order to do this, we move the origin of the coordinate system (x, y) to point $M_1'(x_1, y_1)$ using any available method. Then in a new coordinate system $(\tilde{x}, \tilde{y})$ the distance between points $l = M_1 M_2$ is a length of spherical radius vector of point $M_2'(\tilde{x}, \tilde{y})$.

Let us transform the coordinates, for example, by suing formulas (6):

$$\tilde{x}_2 = \frac{(x_2 - x_1)\sqrt{1 + x_1^2 + y_1^2}}{(1 + x_1 x_2 + y_1 y_2)\sqrt{1 + x_1^2}}, \quad \tilde{y}_2 = \frac{y_2 - y_1 + x_1(x_1 y_2 - x_2 y_1)}{(1 + x_1 x_2 + y_1 y_2)\sqrt{1 + x_1^2}}.$$

Therefore $\mathrm{tg}^2(l/R_0) = \tilde{r}_2^2 = \tilde{x}_2^2 + \tilde{y}_2^2$, or

$$\mathrm{tg}^2 \frac{l}{R_0} = \mathrm{tg}^2\lambda = \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (x_1 y_2 - x_2 y_1)^2}{(1 + x_1 x_2 + y_1 y_2)^2}, \tag{7}$$

where $\lambda \equiv l/R_0$,

If points M_1 and M_2 are spaced apart at a distance $\pi R_0/2$, then $\mathrm{tg}(l/R_0) = \mathrm{tg}(\pi/2) = \infty$, or

$$1 + x_1 x_2 + y_1 y_2 = 0. \tag{8}$$

If point M_2 is located within a short distance from point M_1, so that $x_2 = x_1 + \Delta x$, $\Delta x << 1$; $y_2 = y_1 + \Delta y$, $\Delta y << 1$, then

$$\mathrm{tg}^2\lambda = \frac{\Delta x^2 + \Delta y^2 + (x_1 y_1 + x_1 \Delta y - x_1 y_1 - y_1 \Delta x)^2}{(1 + x_1{}^2 + x_1 \Delta x + y_1{}^2 + y_1 \Delta y)^2} = \frac{\Delta x^2 + \Delta y^2 + (x_1 \Delta y - y_1 \Delta x)^2}{(1 + x_1{}^2 + y_1{}^2 + x_1 \Delta x + y_1 \Delta y)^2}.$$

When $M_2 \to M_1$ we have a definition of measure of length element dl at sphere in tangential coordinate system (x, y) at an arbitrary point $M(x, y)$, matching with definition of measure dl at elliptical plane [6, 7, 17]:

$$\lim \mathrm{tg}^2\lambda = \frac{dl^2}{R_0{}^2} = \frac{dx^2 + dy^2 + (x\,dy - y\,dx)^2}{(1 + x^2 + y^2)^2},$$

i.e.

$$dl^2 = \frac{R_0{}^2}{(1 + x^2 + y^2)^2}\left[dx^2 + dy^2 + (x\,dy - y\,dx)^2\right]. \tag{9}$$

Since $x = r\cos\varphi$, $y = r\sin\varphi$, then $dx = dr\cos\varphi - r\sin\varphi\, d\varphi$, $dy = dr\sin\varphi + r\cos\varphi\, d\varphi$ and from expression (9) we have

$$dl^2 = \frac{R_0^2}{(1 + r^2)^{3/2}}\left[\frac{dr^2}{\sqrt{1 + r^2}} + r^2\sqrt{1 + r^2}\, d\varphi^2\right].$$

The same expression can be deduced from obvious equality $d\lambda^2 = d\rho^2 + \sin^2\rho\, d\varphi^2$.

Let us elementary increment dx and dy tangential coordinates of point M' (Fig.3). Spherical Cartesian coordinates of point M ($\xi = OK/R_0$ and $\eta = OL/R_0$) are also incremented forming an ele-

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mentary area in the sphere $d\tau$ at a point M. In spherical polar coordinate system (ρ, φ) elementary area is $d\tau = \sin\rho\, d\rho\, d\varphi$, and $\sin^2\rho = r^2/(1 + r^2)$.

At the projected plane W: $r = \operatorname{tg}\rho$ and $dr = d\rho/\cos^2\rho = (1 + \operatorname{tg}^2\rho)\, d\rho$. Therefore, in polar-tangential coordinate system (r, φ) the elementary area is

$$d\tau = R_0^2 \frac{r\,dr\,d\varphi}{(1+r^2)^{3/2}}\,. \tag{10}$$

The larger is a distance between point M and origin of coordinate system O, the smaller is elementary area; reduction is proportional to variable $(1 + r^2)^{3/2}$. This is a significant difference between definition of measurement of elliptical plane from Euclidian.

If it remembered that tangential coordinates (x, y) of arbitrary point M are related to its polar-tangential coordinates (r, φ) just as Cartesian and polar ones at Euclidian plane [see expressions (4)], then we have area element $d\tau$ at sphere in tangential coordinates in the form of

$$d\tau = \frac{R_0^2\, dx\, dy}{(1+x^2+y^2)^{3/2}}\,. \tag{11}$$

Formulas (10) and (11) match the definition of measure for element area at elliptical plane [6, 7, 17]. At Euclidian plane $d\tau = dx\, dy$.

Primary spherical curves. Correlation between points and primary curves. Let us consider the main geodesy and navigation tasks for spherical surface of the Earth globe.

Definition of spherical line, similar to definition of a line in Euclidian plane, is a normal form: a line is defined by length p of its normal line from origin of coordinate system and direction of this normal θ (Fig.4).

Let us take an arbitrary point M of this line. Then at the elliptical plane of unit curvature ($R_0 = 1$) we have $r^2 = x^2 + y^2$, $\operatorname{tg}\varphi = y/x$. The angle M_0OM is defined by the correlation

$$\operatorname{tg}(\angle M_0OM) = \operatorname{tg}(\theta - \varphi) = \frac{\operatorname{tg}\theta - \operatorname{tg}\varphi}{1 + \operatorname{tg}\theta \operatorname{tg}\varphi} = \frac{x y_0 - y x_0}{x x_0 + y y_0}\,.$$

From right triangle M_0OM we have

$$r^2 = x^2 + y^2 = \operatorname{tg}^2\rho = \operatorname{tg}^2 p\,[1 + \operatorname{tg}^2(\theta - \varphi)] =$$

$$= \operatorname{tg}^2 p\,(x_0^2 + y_0^2)\,(x^2 + y^2)/(x x_0 + y y_0)^2,$$

or

$$x x_0 + y y_0 = \operatorname{tg}^2 p.$$

However, $x_0 = \operatorname{tg} p \cos\theta$, $y_0 = \operatorname{tg} p \sin\theta$; therefore, for any point M of this line there is a correct equality $x \cos\theta + y \sin\theta = \operatorname{tg} p$, which is a line equation in its normal form. In the limit, when $R_0 \to \infty$, we have a known plane line equation [3, 5] $x \cos\theta + y \sin\theta = p$.

Theorem 4 [13]. Any first-degree equation defines spherical line in tangential coordinates.

Proof. Given first-degree equation for elliptical plane

$$Ax + By + C = 0, \tag{12}$$

where A and B are equal to zero.

Let us divide both parts of the equation (12) by $\sqrt{A^2 + B^2}$ and introduce the symbols:

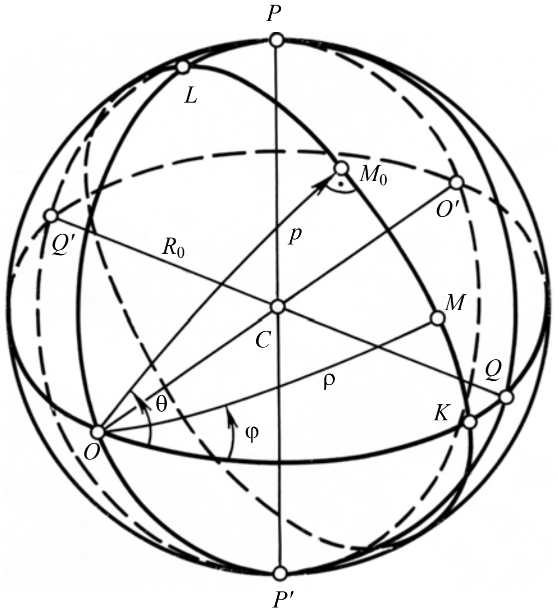

Fig.4. For establishment of normal equation of spherical line

$$\frac{A}{\sqrt{A^2+B^2}}=\cos\theta;\quad \frac{B}{\sqrt{A^2+B^2}}=\sin\theta;\quad \frac{C}{\sqrt{A^2+B^2}}=-\operatorname{tg}p\,.$$

Then equation (12) has a form of normal defined line of a sphere:

$$x\cos\theta+y\sin\theta-\operatorname{tg}p=0,$$

where $\operatorname{tg}\theta=B/A;\ \operatorname{tg}p=-C/\sqrt{A^2+B^2}$.

Notice that if $\operatorname{tg}p\neq 0$, then $\cos\theta/\operatorname{tg}p=\operatorname{ctg}OK/R_0,\ \sin\theta/\operatorname{tg}p=\operatorname{ctg}OL/R_0$, and equation (12) can be transformed into

$$\frac{x}{\operatorname{tg}(OK/R_0)}+\frac{y}{\operatorname{tg}(OL/R_0)}=1 \text{ or } \frac{x}{a}+\frac{y}{b}=1, \tag{13}$$

where $a=\operatorname{tg}(OK/R_0)$; $b=\operatorname{tg}(OL/R_0)$.

Equation (13) is an equation of spherical line in segments at the coordinate axes, which in the limit when $R_0\to\infty$ transforms into similar form of definition of a line at Euclidian plane [3, 5].

It is possible to define a spherical line in the form of $y=kx+c$, где $c=\operatorname{tg}(OL/R_0)$; $k=-\operatorname{ctg}\theta$.

Let us draw a line through a given point $M^*(x^*, y^*)$ of a unit sphere. The line equation will be defined in segments at coordinate axes: $x/a+y/b=1$. If point M^* belongs to this line, then $x^*/a+y^*/b=1$. Therefore, a pencil of lines going through a given point M^*, is defined by the equation $x/a+y/b=x^*/a+y^*/b$, or

$$(x-x^*)/a+(y-y^*)/b=0.$$

Similarly, we can find an equation for a line going through two given points of a sphere: $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$.

An intersection point of spherical lines $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$ is defined like in a case of Euclidian plane. We should remember that any two lines of a sphere always intersect and in two antipodal points: $\xi_0=R_0\operatorname{arctg}x_0$; $\eta_0=R_0\operatorname{arctg}y_0$ и $\xi_0'=R_0(\operatorname{arctg}x_0+\pi)$; $\eta_0'=R_0(\operatorname{arctg}y_0+\pi)$.

If $\Delta=A_1B_2-A_2B_1=0$ and the equality $\Delta_1=\Delta_2=0$ is not kept, the intersection points of spherical lines are on a polar of coordinates origin and defined in polar system of spherical coordinates with the following equalities:

$$\varphi_0=\operatorname{arctg}\Delta_2/\Delta_1;\ \rho_0=\pi R_0/2;\ \ \varphi_0'=\operatorname{arctg}\Delta_2/\Delta_1+\pi;\ \ \rho_0'=\pi R_0/2.$$

If point M_1 is fixed, and position of point M_2 is changed within the condition $M_1M_2=\pi R_0/2$, then from equality (8) we have a polar equation for point M_1

$$\frac{x}{a}+\frac{y}{b}=1,$$

where $a=-1/x_0$; $b=-1/y_0$.

Applying equation (8), we can find coordinates of pole P of this line:

$$x_P=\frac{A}{C}=-\frac{1}{a};\ \ y_P=\frac{B}{C}=\frac{1}{b}. \tag{14}$$

If it is remembered that the angle of intersection ψ of two spherical lines is defined by the distance between their poles, then we can develop a formula

$$\cos^2\psi=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{(A_1^2+B_1^2+C_1^2)(A_2^2+B_2^2+C_2^2)}}. \tag{15}$$

From equality (15) we can derive the condition of orthogonality of spherical lines:

$$A_1A_2+B_1B_2+C_1C_2=0.$$

To draw a normal of this line through an arbitrary point $M_0(x_0, y_0)$, it is sufficient to find a pole of this line and draw a line through it and point M_0. Having done that we will have a normal line equation, drawn from point M_0 to this line: $x(B-Cy_0)++y(Cx_0-A)+(Ay_0-Bx_0)=0$.

Similarly, we can derive the equation of a line going through an arbitrary point $M_0(x_0, y_0)$ and perpendicular to normal line, drawn from point M_0 to this spherical line (the target line at the plane is parallel to a given line):

$$x[x_0(C+By_0)-A(1+y_0^2)]+y[y_0(C+Ax_0)-B(1+x_0^2)]+x_0(A-Cx_0)y_0(B-Cy_0)=0.$$

Distance δ between point $M_0(x_0, y_0)$ and line $Ax + By + C = 0$ can be defined as following. Since this distance equals the complement to $\pi R_0/2$ of distance between point M_0 and pole $P(A/C, B/C)$ of this line, we have

$$\operatorname{tg}\frac{\delta}{R_0}=\operatorname{tg}\left(\frac{\pi}{2}-\frac{M_0P}{R_0}\right)=\operatorname{ctg}\frac{M_0P}{R_0}=\frac{1}{\operatorname{tg}(M_0P/R_0)},$$

i.e.

$$\operatorname{tg}^2\frac{\delta}{R_0}=\frac{(Ax_0+By_0+C)^2}{(A-Cx_0)^2+(B-Cy_0)^2+(Bx_0-Ay_0)^2}.$$

Main parameters of secondary and higher-degree curves. The previously developed analytical apparatus enables compiling ellipse equation with focuses at points $F_1(\delta, 0)$ and $F_2(-\delta, 0)$ of a unit sphere and with major semi-axes α and minor semi-axis β. The detailed development of this equation is published in article [2], where we derived a canonical equation for ellipse at the sphere in the form of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{16}$$

where $a = \operatorname{tg}\alpha$, $d = \operatorname{tg}\delta$, $b^2 = \operatorname{tg}^2\beta = (a^2 - d^2)/(1 + d^2)$.

The canonical equation of hyperbola at the sphere can be derived from ellipse equation (16) using the transformation of coordinates (5):

$$\frac{x^2}{g^2}-\frac{y^2}{q^2}=1,$$

where $g = 1/a = \operatorname{tg}\zeta$; $q = b/a$; ζ is a hyperbola parameter.

Using theorem 3 and transformed coordinates (5), we develop canonical equation of spherical parabola $y^2 = 2Px$, where $P = \sin p$; p – parabola parameter.

Circle equation, which center is set at the coordinates origin, helps to define in correspondence with formula (4): $x^2 + y^2 = r_0^{\,2} = \text{const}$, where $r_0 = \operatorname{tg}\rho_0$; ρ_0 – a radius of spherical circle.

Equations of tangent to secondary curves at a certain point of a sphere (x_0, y_0) are used to define limits of the corresponding secant lines. For example, for spherical hyperbola

$$\frac{xx_0}{g^2}-\frac{yy_0}{h^2}=1.$$

The secondary curves can be defined in parametric form. Hence, by substituting into equation (16) we can check that for a canonical ellipse the parametric equation has a form of $x = a\cos\varphi$; $y = b\sin\varphi$. And parameter φ ranges from 0 to 2π.

Assume for our convenience that major ellipse semi-axis a is directed along axis Oy. Therefore, $x = b\cos\varphi$, $y = a\sin\varphi$ and

$$dx=-b\sin\varphi\, d\varphi;\ dy=a\cos\varphi\, d\varphi, \tag{17}$$

and linear element of spherical ellipse is derived by substituting expressions (17) into equality (9):

$$dl^2=R_0^2\,\frac{b^2\sin^2\varphi+a^2\cos^2\varphi+(ab\cos^2\varphi+ab\sin^2\varphi)^2}{(1+b^2\cos^2\varphi+a^2\sin^2\varphi)^2}\,d\varphi^2.$$

Taking into account, that $\cos^2\varphi = 1 - \sin^2\varphi$, after identity transformations we have

$$dl^2 = R_0^2 a^2 \frac{1-k^2\sin^2\varphi}{(1+b^2)(1+h\sin^2\varphi)^2} d\varphi^2,$$

where $k^2 = (a^2 - b^2)/[a^2\,(1 + b^2)]$; $h = (a^2 - b^2)/(1 + b^2) = k^2a^2$.

The length element dl of the ellipse can be written as

$$dl = R_0 \frac{1+a^2-(1+h\sin^2\varphi)}{a\sqrt{1+b^2}\,(1+h\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}} d\varphi .$$

Therefore, the length of the sector $(0, \theta)$ of spherical ellipse is calculated using formula

$$L_{\text{с.э}}(\theta) = \frac{R_0(1+a^2)}{a\sqrt{1+b^2}} \int\limits_0^\theta \frac{1}{(1+h\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}} d\varphi - \frac{R_0}{a\sqrt{1+b^2}} \int\limits_0^\theta \frac{1}{\sqrt{1-k^2\sin^2\varphi}} d\varphi$$

or

$$L_{\text{с.э}}(\theta) = \frac{R_0(1+a^2)}{a\sqrt{1+b^2}} \Pi(\theta, h,\ k) - \frac{R_0}{a\sqrt{1+b^2}} F(\theta, k),$$

where $F(\theta, k)$ and $\Pi(\theta, h, k)$ are reduced elliptic integrals of I and III types accordingly [11, 14].

The full length of spherical ellipse $L_{\text{с.э}}$ is calculated by integrating length element dl by parameter φ from 0 to 360° and has four equal-sized semicircular arcs:

$$L_{\text{с.э}} = \frac{4R_0(1+a^2)}{a\sqrt{1+b^2}} \Pi\left(\frac{\pi}{2}, h,\ k\right) - \frac{R_0}{a\sqrt{1+b^2}} \mathbf{K}(k), \tag{18}$$

where $\mathbf{K}(k) = F(\pi/2, k)$ is a full elliptical integral of I type [10, 11, 14].

For the first time this formula (18), apparently, was developed in 1958 by Nguen Kan Toan [9].

The area of the spherical ellipse $S_{\text{с.э}}$ can be found using integration of expression (11) with variables x and y:

$$S_{\text{с.э}} = 4R_0^2 \int\limits_0^a \int\limits_0^{y(x)} \frac{dy}{(1+x^2+y^2)^{3/2}} dx .$$

As a result we will have [9]

$$S_{\text{с.э}} = \frac{4R_0\, b}{a\sqrt{1+a^2}} \Pi\left(\frac{\pi}{2}, h',\ k'\right) - \frac{4R_0}{a\sqrt{1+b^2}} \mathbf{K}(k'),$$

where $k'^2 = (a^2 - b^2)/(1 + a^2)$; $h' = -a^2/(1 + a^2)$.

Using parametric definition of curves of higher-degree we can similarly develop expressions for length and area of more complex geometrical figures in sphere. For example, Archimedean spiral at unit sphere in polar spherical coordinates is defined with equation $\rho = a\,\varphi$, $-\infty < \varphi < \infty$. Therefore, $d\rho = a\,d\varphi$, and length element dl of this spiral is described with equalities

$$dl^2 = d\rho^2 + \sin^2\rho\, d\varphi^2 = [a^2 + \sin^2(a\,\varphi)]\, d\varphi^2.$$

The length of the spiral segment as a function of variable θ with $R_0 \neq 1$

$$L_{\text{а.с}}(\theta) = \frac{R_0\sqrt{1+a^2}}{a} E\left(\mu, \frac{1}{\sqrt{1+a^2}}\right) - \frac{R_0}{a} \frac{\sin x\cos x}{\sqrt{a^2+\sin^2 x}}, \tag{19}$$

where $E(\mu, k)$ is not full elliptical integral of II type; $k \equiv 1\big/\sqrt{1+a^2}$; $x \equiv a\,\theta$; $\mu \equiv \arcsin\left(\sqrt{1+a^2}\sin x\big/\sqrt{a^2+\sin^2 x}\right)$ [10, 11, 14].

Conclusion. When introducing tangential coordinates at spherical surface the mathematical tools of analytical geometry for a sphere is a little bit more difficult then classical analytical geometry for Euclidian plane. At the same time a known set of tools for spherical trigonometry is simpler than the tools of analytical geometry of a sphere. All it takes is to compare the equation for a line crossing two given points in tangential coordinates

$$(x-x_1)/(x_2-x_1)=(y-y_1)/(y_2-y_1)$$

and geographical coordinates

$$\operatorname{tg}\varphi'\sin\operatorname{tg}\varphi'\sin(\lambda'_2-\lambda'_1)+\operatorname{tg}\varphi'_1\sin(\lambda'-\lambda'_2)+\operatorname{tg}\varphi'_2\sin(\lambda'_1-\lambda')=0\,,$$

where (x_1, y_1) and (x_2, y_2) are tangential coordinates of points M_1 and M_2 at the unit sphere; (φ'_1, λ'_1) and (φ'_2, λ'_2) are geographical coordinates of the same points.

It is also difficult to define the length and area of different figures at the sphere surface using geographical coordinates. Using tangential coordinates these variables are often transformed into known special functions.

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