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SPATIAL DISTRIBUTION OF ENERGY RELEASE DURING PROPAGATION OF FAST ELECTRON BEAM IN THE AIR

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The paper focuses on development of the analytical theory to assess spatial distribution of energy released during propagation of the fast electron beam in a gas, in particular in the air at electron energies of 1-100 keV. An approach adopted by authors [2, 3] to study inelastic deceleration of electrons in the air is further developed here. As the inelastic interaction in most cases leads to energy relaxation while elastic interaction causes distribution isotropization over directions, the first task solved in the paper is finding the electron distribution function including only elastic collisions. In the final part of this paper an analytical solution to this task is presented with account of both types of electron deceleration in the air.

The calculations show that when elastic collisions are taken into account this leads to increased spatial density of energy release and to narrowing of the primary energy release region of the fast electrons, as compared to calculations accounting for only inelastic deceleration.

Key words: deceleration of electron beam with energy of 1 to 100keV in a gas, elastic and inelastic interactions of electrons, energy and momentum relaxation, electron velocity distribution function, Boltzmann kinetic equation, numerical simulation based on Monte Carlo method.

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Introduction For hypersonic aircraft engineering actual relaxation of fast electrons with energies up to 100 keV in a gas is a task of high relevance. In order to optimize the external aerodynamic flows and improve engine performance of such aircrafts it is proposed to use MHD control of gas flows. For this purpose in order to create MHD effect a high-energy electronic beam can be applied. Calculation of the ionization degree of emerging plasma is reduced to calculation of the spatial distribution of energy released in the gas. This distribution in its turn is simulated using the Monte Carlo method. As the initial energies of electrons are high, while differential cross-section of electron collisions with atoms and gas molecules is rapidly decreasing with increase in energy transferred to the atom, the loss of electron energy in each collision is minor. This results in a need to take into account dozens of thousands of collisions, which even given the present-day high speed computing is very time consuming.

This paper focuses on development of the analytical theory to assess spatial energy distribution of fast electron beam with energies from 1 to 100 keV during its propagation in the air. An approach adopted by authors [2, 3] to study inelastic deceleration of electrons in the air is further developed here.

First the task of finding the electron distribution function including only elastic collisions is solved. In the final part of this paper an analytical solution to this task is presented with account of both types of electron deceleration in the air.

1. Solution of Boltzmann equation including only elastic collisions. Let's use a physical model developed in study [2]. Besides let's use the same notations as used in the above study. Here it should be reminded that we review an approximation when inelastic collisions of electrons with gas molecules lead only to changes in energy while elastic collisions cause changes in direction. As demonstrated in [2], such approach is reasonable in experiment conditions. Differential cross-section of electron elastic scattering by molecules, mentioned in [8], can be described with the following formula

$$\frac{d\sigma_e}{d\Omega} = \frac{r_0^2 Z^2 (1-\beta^2)}{\beta^4 (1-\cos\theta + 2\eta)^2}, \qquad (1)$$

where r_0 is the classical radius of electron; $\alpha = Z/137; \beta = v/c$; Z, v and c are the charge of target atom nucleus, the electron velocity and the speed of light, respectively; and Θ is the polar scattering angle.

The screening parameter in Moliere approximation is defined based on the following correlation [8]

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$$\eta = 1.7 \cdot 10^{-5} Z^{\frac{2}{3}} \frac{(1-\beta)}{\beta^2}\left(1.33 + 3.76\frac{\alpha^2}{\beta^2}\right). \quad (1a)$$

The Boltzmann equation including only elastic collisions is as follows

$$\mu\frac{\partial f}{\partial z} + \Lambda_e(E) f = 2\pi N \int_0^{2\pi}\int_{-1}^{1} f(E,\mu',z)\sigma_e(E,\mu_0)d\mu' d\varphi, \quad (2)$$

where $\Lambda_e(E)$ is the inverse free path for elastic collisions; $\mu_0 = \cos\theta = = \mu\mu' + \sqrt{1-\mu'^2}\sqrt{1-\mu^2}\cos\varphi$ is the cosine of shattering angle.

As the distribution function does not depend on the azimuthal angle φ, when performing integration with this variable in the right-hand side (2), we obtain the following

$$\mu\frac{\partial f}{\partial \tilde{z}} + f = \int_{-1}^{1} f(E,\mu',z) g_e(E,\mu' \to \mu) d\mu', \quad (3)$$

where $\tilde{z} = z\Lambda_e(E)$; $g_e(E,\mu' \to \mu) = \dfrac{2A(\eta)(1+2\eta-\mu\mu')}{\{(\mu-\mu')^2 + 4\eta(1+\eta-\mu\mu')\}^{1,5}}$;

$$A(\eta) = \eta(1+\eta);\ \Lambda_e(E) = \pi r_0^2 Z^2 \frac{(1-\beta^2)}{A(\eta)\beta^4}.$$

Therewith normalization condition is met

$$\int_{-1}^{1} g_e(E,\mu' \to \mu)d\mu' = \int_{-1}^{1} g_e(E,\mu' \to \mu)d\mu = 1.$$

Let's have a look at the model scattering indicatrix $g_{me}(E,\mu' \to \mu)$:

$$g_{me}(E,\mu' \to \mu) = a_1(E)\delta(\mu'-\mu) + a_2(E). \quad (4)$$

Let's select $a_1(E), a_2(E)$ functions based on condition of equity of two first moments of model and accurate indicatrix. Thus we obtain

$$a_2 = \frac{1-a_1}{2};\ a_1 = \frac{\bar{\mu}}{\mu}, \quad (5)$$

where $\bar{\mu} = \int_{-1}^{1} g_e(E,\mu' \to \mu)\mu' d\mu'$, and coefficients $a_1(E), a_2(E)$ appear to be non-dependent on μ.

By substituting indicatrix $g_{me}(E,\mu' \to \mu)$ in equation (3) and using the first of the equalities (5), we obtain

$$\mu\frac{\partial f}{\partial z'} + f = \frac{1}{2}\int_{-1}^{1} f(z',\mu')d\mu' \text{ at } z' = 2a_2\tilde{z}. \quad (6)$$

Thus, we have reduced the problem to the kinetic Boltzmann equation with a spherically symmetric scattering indicatrix. Equation (6) shall be complemented by a boundary condition

$$f(z',\mu)_{\substack{z=0 \\ \mu>0}} = \delta(\mu-1). \quad (6a)$$

To solve task (6), (6a) results presented in [3, 4] can be used. According to these results solution to equation (3), restricted to infinity, can be presented as follows:

$$f(z',\mu) = A_0 + \int_0^1 A(u)\exp\left(-\frac{z'}{u}\right)G_u(\mu)du. \quad (7)$$

Using the boundary condition (6a) we may obtain

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$$A_0 = 5.036;\ A(u) = 1.679\frac{\left\{P_k\left[\frac{u}{u-\mu}\right] + p(u)\delta(u-1)\right\}}{C(u)P(u)}, \tag{8}$$

where functions, included in (7), are defined by formulas [4,5]:

$$A_0 = 3\int_0^1 \mu P(\mu) f(0,\mu) d\mu;$$
$$A(u) = \left[C(u)P(u)\right]^{-1}\int_0^1 \mu P(\mu) f(0,\mu) G_u(\mu) d\mu; \tag{9}$$

$$C(u) = \frac{u}{2}\left\{p^2(u) + \pi^2 u^2\right\} \text{ at } p(u) = 2 + \ln\left(\frac{1-u}{1+u}\right);$$

$$P(\mu) = \mu \exp\left(-\frac{1}{\pi}\int_0^1 \operatorname{arctg}\left(\frac{\pi t}{p(t)}\right)\frac{dt}{t+\mu}\right),$$

and $\operatorname{arctg}(\pi t)$ changes from $-\pi$ to 0, when t changes from 0 to 1.

Function $G_u(\mu)$ can be defined using formula

$$G_u(\mu) = P_k\left[\frac{u}{u-\mu}\right] + p(u)\delta(u-\mu).$$

Expression $P_k\left[\frac{u}{u-\mu}\right]$ is regarded as a generalized function determined with the help of the integral composite function:

$$\int_{-1}^{1} P_k\left[\frac{u}{u-\mu}\right] F(u)du = \lim_{\varepsilon\to 0}\left(\int_{-1}^{\mu-\varepsilon}\frac{uF(u)}{u-\mu}du + \int_{\mu+\varepsilon}^{1}\frac{uF(u)}{u-\mu}du\right)$$

with arbitrary function $F(u)$.

Omitting cumbersome arithmetics below are the end formulas for $N(z'), \overline{\mu}_+(z'),\ \overline{\mu}_-(z'), \overline{\mu}_+^{\,2}(z')$ values:

$$N(z') = 2A_0 + B_1(z') + C_1(z');\ \overline{\mu}_+(z') = \frac{0.5A_0 + A_1(z')}{A_0 + B_1(z')};$$
$$\overline{\mu}_-(z') = \frac{-0.5A_0 + D_1(z')}{A_0 + C_1(z')};\ \overline{\mu}^2(z') = \frac{2A_0}{3\left[2A_0 + B_1(z') + C_1(z')\right]}; \tag{10}$$
$$\overline{\mu}_+^{\,2}(z') = \frac{\frac{A_0}{3} + G_1(z')}{A_0 + B_1(z')}.$$

While

$$A_1(z') = \int_0^\infty F_0(y)u(y)\exp\left(-\frac{z'}{u(y)}\right)dy;\ B_1(z') = \int_0^\infty F_1(y)\exp\left(-\frac{z'}{u(y)}\right)dy;$$

$$C_1(z') = \int_0^\infty F_2(y)\exp\left(-\frac{z'}{u(y)}\right)dy;\ D_1(z') = \int_0^\infty F_3(y)\exp\left(-\frac{z'}{u(y)}\right)dy; \tag{11}$$

$$G_1(z') = \int_0^\infty F_4(y)\exp\left(-\frac{z'}{u(y)}\right)dy;\ u(y) = 1 - \exp(-y),$$

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and $F_0(z'), F_1(z'), F_2(z'), F_3(z')$ functions are calculated by the following formulas:

$$F_0(z') = -1.679\frac{\left[p_1(z') - 1 + u(z')\ln\left(u(z')\exp(z')\right)\right]u(z')}{P(u(z'))C(u(z'))};$$

$$F_1(z') = -1.679\frac{\left[p_1(z') + u(z')\ln\left(u(z')\exp(z')\right)\right]u(z')}{P(u(z'))C(u(z'))};$$

$$F_2(z') = 1.679\frac{u^2(z')\ln\left(u(z')\exp(z')\right)}{P(u(z'))C(u(z'))}; \qquad (12)$$

$$F_3(z') = 1.679\frac{\left[1 + u(z')\ln\left(u(z')\exp(z')\right)\right]u^2(z')}{P(u(z'))C(u(z'))};$$

$$F_4(z') = -1.679\frac{\left[u^2(z')\ln\left(u(z')\exp(z')\right) - u(z') - 0.5 + u(z')p_1(z')\right]u^2(z')}{P(u(z'))C(u(z'))};$$

$$p_1(z') = 2 - u(z')\left[z' + \ln(1 + u(z'))\right].$$

Besides the following correlation is reached

$$D_1(z') - A_1(z') \equiv A_0. \qquad (13)$$

Formulas (7), (8), (9)-(13) provide solution to the task in hand. For specific calculations of function $f(z', \mu)$ moments the following approximation formula appears to be useful with a span of error of a few percentage points:

$$P(u) \approx P_1(u) = 1.676 - 1.095\left(1 - u\right)^{1,1}. \qquad (14)$$

It is apparent that transition from variable z' (distance expressed in path lengths calculated by isotropic part of elastic scattering cross-section) to variable $\tilde{z}$ (distance expressed in path lengths calculated by full elastic scattering cross-section) is performed using the following formula

$$z' = \tilde{z} \cdot 2a_2(E) = \tilde{z}\left(1 - \frac{\bar{\mu}(E)}{\mu}\right). \qquad (15)$$

Besides for the numerical calculations the following approximation is useful with accuracy of several percentage points:

$$a_2(E) \approx \bar{a}_2(E) = \left[1 + 0.057\frac{E^2}{1 + 0.057E}\right]^{-1}. \qquad (16)$$

2. Complete solution to the problem of high energy electron beam propagation in the air. As demonstrated by calculations [2], in the electron energy range being of interest for us the inelastic scattering cross-section greatly exceeds the elastic scattering cross-section. Thus when energy is 2 keV the correlation is about 20 and with energy increase it increases further, since, as mentioned previously, the inelastic scattering cross-section as opposed to elastic scattering cross-section has almost no dependence on energy. So between relatively rare elastic collisions, which change the direction of electrons, the electrons move in straight lines, experiencing inelastic collisions in which the energy is lost.

Let's first consider the effect of energetic relaxation between successive elastic collisions on the results obtained. In order to measure this effect we assume that the electrons experience only elastic collisions, but before each collision their energy decreases and becomes equal to the mean energy of the flow $E(\bar{z})$, calculated only with account of inelastic collisions in the study [2]. While all distances will be expressed in the length of path between inelastic collisions, i.e. thus we move to variable $z_0(E_0) > \bar{z}$. It is obvious, that spatial variable in all formulas shall now take $X(\bar{z})$ value:

$$X(\bar{z}, E_0) = \frac{2\sigma_e(E_0)}{\sigma_n(E_0)}\int_0^{\bar{z}} a_2\left[E(z_1)\right]dz_1. \qquad (17)$$

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When calculating integral in this formula we obtain the following

$$X(\bar{z},E_0)=-\frac{1}{K(E_0)}\left[\frac{T_1(\bar{z},E_0)}{2}+4.098T(\bar{z},E_0)\right];$$
$$T_1(\bar{z},E_0)=\operatorname{arctg}[0.12(2E_0+1)]-\operatorname{arctg}\left[0.12\left(2E_0t(\bar{z},E_0)\right)\right];$$
$$T(\bar{z},E_0)=\ln\left[\frac{1+0.057E_0(1+E_0)}{1+0.057E_0t(\bar{z},E_0)\left(1+E_0t(\bar{z},E_0)\right)}\right]; \quad (17a)$$
$$t(\bar{z},E_0)=1+\bar{z}\frac{K_0(E_0)}{E_0}.$$

Let's calculate $\bar{z}$ values at which complete isotropization of angular electron flow propagation occurs, i.e. when the below correlation is reached

$$\bar{\mu}_+[X(\bar{z})]\approx\bar{\mu}_-[X(\bar{z})]\approx\frac{1}{2}.$$

With account of the above results we see that the respective value of $\bar{E}_1(\bar{z},E_0)$ can be found as a solution to transcendent equation

$$X[z_0(E_0)]=4. \quad (18)$$

$z_0(E_0)$ value turns to be much greater than 1 and, for example, when $E_0=2\,\text{keV}$ the following is true $z_0\cong 90$, and for $E_0=100\,\text{keV}$ this parameter is $z_0\cong 3.8\cdot10^4$.

In study [2] it has been shown that the following dependency $E(\bar{z})$ can be well illustrated by a linear function. Therewith almost full relaxation of energy corresponds to the following distance from the departure surface:

$$z_e(E_0)=\frac{-E_0}{K(E_0)}, \quad (19)$$

with function $K(E_0)$ defined in [2].

Calculations of $r(E_0)=z_0(E_0)/\bar{z}_e(E_0)$ variable show that at initial energy of electrons $2\,\text{keV}<E_0<100\,\text{keV}$ its value is in the range between 18 and 35%. While the estimated mean electron energy in region $\bar{z}<z_0(E_0)$ with account of other inelastic collisions indicates that its difference from the initial energy does not exceed 0.2%. It means that isotropization of angular electron beam distribution occurs within the indicated energy range with virtually no relaxation of electrons' energy and at a distances from the departure surface much smaller than the region of major energy release. Then the following solution to the task can be suggested with account of inelastic and elastic collisions of electrons with gas molecules.

Let's first review region $\bar{z}>z_0(E_0)$. As we could see at $\bar{z}=z_0(E_0)$ distribution of electrons is almost isotropic. Then electrons experience inelastic collisions coupled with loss of energy and elastic ones changing their direction. But then it can be easily proved that at any $\bar{z}>z_0(E_0)$ electron distribution function is isotropic. In such a case it is obvious that elastic collisions which change only direction and do not lead to any changes in distribution function may be neglected. Besides within the frames of the model proposed in [2] we assume that full cross-section of inelastic collisions has almost not dependence on the electron energy In this case kinetic equation for electron distribution function looks as follows

$$\mu\frac{\partial f(\bar{z},t,\mu)}{\partial\bar{z}}+f(\bar{z},t,\mu)=a\int_0^t f(\bar{z},t',\mu)\sigma_{nd}(t-t')dt', \quad (20)$$

where μ is the cosine of an angle between the inner normal to the plane and the electron direction;

$t=1-x;\ x=E/E_0;\ \ a=1/\sum_n(1,E_0);\ \sum_n(x,E_0)=E_0\int_0^x\sigma_{nd}(x,x-x')dx'.$

As it was demonstrated in study [1] in this situation in order to solve kinetic equation the so called P_1-approximation can be used. Based on the same assumption as made in study [2], that the cross-section $\sigma_{nd}(t-t')$ has an exponential form:

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$$\sigma_{nd}(t-t') = \Sigma_n(1,E_0)b(E_0)\exp\left[-b(E_0)(t-t')\right], \tag{20a}$$

(formulas for values included in (20a) are presented in [2]), let's present $f(\bar{z},t,\mu)$ in the following form

$$f(\bar{z},t,\mu) = \frac{1}{2}\left[f^{(0)}(\bar{z},t) + 3\mu f^{(1)}(\bar{z},t)\right]. \tag{21}$$

Let's put (21) into (20) and integrate it by all electron directions, then put (21) in (20), multiply by μ and also integrate it. In the end we obtain the following system of linear integrodifferential equations for the functions $f^{(0)}(\bar{z},t), f^{(1)}(\bar{z},t)$:

$$\begin{aligned} &\frac{\partial f^{(1)}(\bar{z},t)}{\partial \bar{z}} + f^{(0)}(\bar{z},t) = a\int_0^t f^{(0)}(\bar{z},t')\sigma_{nd}(t-t')dt';\\ &\frac{1}{3}\frac{\partial f^{(0)}(\bar{z},t)}{\partial \bar{z}} + f^{(1)}(\bar{z},t) = a\int_0^t f^{(1)}(\bar{z},t')\sigma_{nd}(t-t')dt'. \end{aligned} \tag{22}$$

Boundary condition for system (22) can be written down as

$$f^{(0)}(z_0,t) = \Phi_0(t);\ f^{(1)}(z_0,t) = \Phi_1(t), \tag{23}$$

where $\Phi_0(t), \Phi_1(t)$ stands for some yet unknown functions.

Excluding function $f^{(0)}(z,t)$ out of the system (22) and applying to the resulting equation the Laplace transform using t variable, we obtain the Laplace's equation $g_1(z,p)$ for function $f^{(1)}(z,t)$:

$$\frac{\partial^2 g_1}{\partial z^2} - 3\left[1-al(p)\right]^2 g_1 = 0;\ g_1(z_0,p) = G_1(p), \tag{24}$$

where $G_1(p)$ and $l(p)$ are the Laplace transforms for functions $\Phi_1(t)$ and $\sigma_{nd}(t)$ respectively.

Assuming that cross-section $\sigma_{nd}(t)$ is determined by formula (20a) and applying the Laplace inverse transform to the solution of equation (24), with account of (23) we get

$$\begin{aligned} f^{(1)}(\bar{z},t)_{z>z_0} = {}& \Phi_1(t)\exp\{-\sqrt{3}(\bar{z}-z_0)\} + \exp\{-\sqrt{3}(\bar{z}-z_0)\}\sqrt{a(\bar{z}-z_0)}\times\\ &\times\int_0^t \Phi_1(t-y)I_1\left(2\sqrt{a(\bar{z}-z_0)y}\right)\frac{\exp(-ay)}{\sqrt{y}}dy. \end{aligned} \tag{25}$$

Now let's review region $z_0(E_0) > \bar{z}$. Integrating Boltzmann equation tor electron distribution function with account of both inelastic and elastic deceleration [2] for region $z_0(E_0) > \bar{z}$ the following system of equations is obtained:

$$\begin{aligned} &\frac{\partial f^{(1)}(\tilde{z},t)}{\partial \bar{z}} + f^{(0)}(\bar{z},t) = b(E_0)\int_0^t f^{(0)}(\bar{z},t')\sigma_{nd}(t-t')dt';\\ &\frac{\partial[\bar{\mu}^2(\bar{z})f^{(0)}(\bar{z},t)]}{\partial \bar{z}} + f^{(1)}(\bar{z},t) = b(E_0)\int_0^t f^{(1)}(\bar{z},t')\sigma_{nd}(t-t')dt', \end{aligned} \tag{26}$$

where $\bar{\mu}^2(z) = \int_{-1}^{1}\mu^2 f(z,0,\mu)d\mu \Big/ \left(\int_{-1}^{1} f(z,0,\mu)d\mu\right)$.

Performing the same sequence of steps as described above we obtain

$$\frac{\partial^2[\bar{\mu}^2(\bar{z})g_0]}{\partial \bar{z}^2} - \left[1-b(E_0)l(p)\right]^2 g_0 = 0;\ g_0(z_0,p) = G_0(p), \tag{27}$$

where $G_0(p)$ is the Laplace transform for function $\Phi_0(t)$.

As illustrated by estimates the equation (27) meets the conditions of WKB-approximation [6]. In this approximation solution to equation (27) looks as follows

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$$g_0(\bar{z},p)=G_0(p)\frac{\sqrt{\bar{\mu}^2(0)}}{\sqrt{\bar{\mu}^2(\bar{z})}}\exp\{-\bar{z}\}\exp\left\{b(E_0)l(p)\int_0^{\bar{z}}\frac{dy}{\sqrt{\bar{\mu}^2(y)}}\right\}. \quad (28)$$

From (26) similarly to the case $z_0(E_0)<\bar{z}$ we obtain

$$g_1(\bar{z},p)=G_1(p)\exp\{-(\bar{z}-z_0)\}\exp\left\{b(E_0)l(p)\int_{z_0}^{\bar{z}}\frac{dy}{\sqrt{\bar{\mu}^2(y)}}\right\}.$$

Applying the Laplace inverse transform for region $z_0(E_0)>\bar{z}$ we finally obtain

$$f^{(1)}(\bar{z},t)_{\bar{z}<z_0}=\Phi_1(t)\exp\left\{-\int_{z_0}^{\bar{z}}\frac{dz}{\sqrt{\bar{\mu}^2(z)}}\right\}+\exp\left\{-\int_{z_0}^{\bar{z}}\frac{dz}{\sqrt{\bar{\mu}^2(z)}}\right\}\times$$

$$\times\sqrt{b(E_0)(\bar{z}-z_0)}\int_0^t\Phi_1(t-y)I_1\left(2\sqrt{b(E_0)(\bar{z}-z_0)y}\right)\frac{\exp(-b(E_0)y)}{\sqrt{y}}dy. \quad (29)$$

Here it's worth reminding that the purpose of this study is to obtain the following values $\bar{E}(\bar{z},E_0)$; $\frac{d\bar{E}(\bar{z},E_0)}{d\bar{z}}$. Therewith in addition to the distribution function integration by variable E (with E weight) the integration shall be performed by all electron directions with a weight equal to μ. It is obvious, than when distribution function is determined by expression (21), a summand that includes $f^{(0)}$ will turn to zero, that's why there is no need to calculate a symmetrical part of electron distribution function.

Now the function shall be found which according to definition looks as follows

$$\Phi_1(t)=\int_{-1}^{0}\mu\cdot f(0,t,\mu)d\mu+\delta(t)\exp\left\{-\int_0^{z_0}\frac{dz}{\sqrt{\bar{\mu}^2(z)}}\right\}. \quad (29a)$$

Let's show that in the range with electrons' initial energies $E_0>2$ keV the first summand in expression for $\Phi_1(t)$ (29a) can be neglected. From the above it is clear that the flow of electrons moving from the half-space to the departure plane is weakly anisotropic. Then in kinetic equation at $\mu<0$ elastic collisions may be neglected. Electrons moving towards the departure surface appear due to changes in electron directions after elastic collisions. It is clear that the smaller is the value of $\mu>0$ before the elastic collision, the greater is the probability that this electron after the collision will change its direction and move towards the departure plane, i.e. that after collision it will have $\mu<0$. As we would like to make an assessment from the top, we assume, that all electrons with $\mu>0$ are moving with the minimum possible cosine between the normal to the departure plane and the electron direction, i.e. with $\mu=0.5$, corresponding to isotropic distribution. Integrating kinetic equation by μ in a range from 0 to -1 with masses 1 and μ, we obtain the following system of equations:

$$\begin{aligned}&\frac{\partial f_-^{(1)}(\bar{z},t)}{\partial\bar{z}}+f_-^{(0)}(\bar{z},t)=b(E_0)\int_0^t f_-^{(0)}(\bar{z},t')\sigma_{nd}(t-t')dt'+\alpha_1\frac{\sigma_e(E_0)}{\sigma_n(E_0)}f_+(\bar{z},t);\\&\frac{1}{3}\frac{\partial f_-^{(0)}(\bar{z},t)}{\partial\bar{z}}+f_-^{(1)}(\bar{z},t)=b(E_0)\int_0^t f_-^{(1)}(\bar{z},t')\sigma_{nd}(t-t')dt'+\alpha_0\frac{\sigma_e(E_0)}{\sigma_n(E_0)}f_+(\bar{z},t),\end{aligned} \quad (30)$$

where $\alpha_1=\int_{-1}^{0}g_e\left(E_0,\frac{1}{2}\to\mu,1\right)\mu d\mu$; $f_+(\bar{z},t)$ is calculated by formula (23a) presented in study [2]; $\alpha_0=\int_{-1}^{0}g_e\left(E_0,\frac{1}{2}\to\mu\right)d\mu$.

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Applying to system (30) the Laplace transform and excluding function $g_-^{(0)}$, we obtain the second order differential equation for function $g_-^{(1)}$. Solution to this equation looks as follows

$$g_-^{(1)} = \frac{1}{4\beta(p)} \frac{\sigma_e(E_0)}{\sigma_n(E_0)} (\alpha_0 - 3\alpha_1) \exp[\beta(p)\bar{z}], \tag{31}$$

where $g_-^{(1)}$ is the Laplace transform for function $f_-^{(1)}(\bar{z},t)$; $\beta(p) = 1 - b(E_0)l(p)$.

When $l(p)$ is determined by formula $l(p) = b(E_0)/[p + b(E_0)]$, application of the Laplace inverse transform yields

$$f_-^{(1)}(0,t) = \frac{1}{4} \frac{\sigma_e(E_0)}{\sigma_n(E_0)} (\alpha_0 - 3\alpha_1)[\delta(t) + b(E_0)]. \tag{32}$$

Therefore for this energy flow from the half-space $\bar{z} > 0$ through the departure plane we have the following

$$E_-(0) = \frac{E_0}{4} \frac{\sigma_e(E_0)}{\sigma_n(E_0)} (\alpha_0 - 3\alpha_1) \left[1 + \frac{b(E_0)}{2}\right]. \tag{33}$$

Calculations show that the value of $E_-(0)/E_0$ at $E_0 = 2$ keV is a few percent of E_0 and decreases rapidly with the increase in electron initial energy. Physically it can be explained, firstly, by relatively small scattering angle at the time of elastic collisions, and secondly, by relatively small cross-section of elastic scattering as compared to the cross-section of inelastic scattering. Thus, when calculating energy release in a gas at $\bar{z} = 0$ the distribution function of electrons moving towards the departure plane may be neglected. Then the below correlation is reached with the accuracy as specified

$$\Phi_1(t) = \delta(t) \exp\left\{-\int_0^{z_0} \frac{dz}{\sqrt{\bar{\mu}^2(z)}}\right\}. \tag{34}$$

Using formulas (29) and (34) for mean energy and energy release within the value range $z_0(E_0) > \bar{z}$ we can obtain the following correlations:

$$\bar{E}_{ne}(\bar{z}) = \bar{E}_n(Y(\bar{z}, E_0)); \tag{35}$$

$$\frac{d\bar{E}_{ne}(\bar{z})}{d\bar{z}} = \frac{1}{\sqrt{\bar{\mu}^2(X(\bar{z}, E_0))}} \frac{d\bar{E}_n(Y(\bar{z}, E_0))}{dY};$$

$$Y(\bar{z}, E_0) = \int_0^{\bar{z}} \frac{dz}{\sqrt{\bar{\mu}^2[X(z, E_0)]}},$$

where $X(\bar{z}, E_0)$ is calculated by formulas (17a) and (18) of this paper and values $\bar{E}_n(\bar{z})$ and $\frac{d\bar{E}_n(\bar{z})}{d\bar{z}}$ by formulas (24) and (25) respectively of paper [2].

For numeric calculations in order to reduce the time of calculations it's convenient to use the following approximation for function $\bar{\mu}^2(z)$:

$$\bar{\mu}^2(z) = \left\{0.578\left[1 + \frac{0.313}{(1 + 0.55z)^{4.2}}\right]\right\}^2.$$

It shall be noted that as at $z_0(E_0) < \bar{z}$ the below correlation is reached within a tens of a percent

$$\sqrt{\bar{\mu}^2(X(\bar{z}, E_0))} = \frac{1}{\sqrt{3}},$$

the formulas (35) are valid for the entire half-space. It should be remembered that formulas (35) of the present study give a possibility to take elastic collisions into account, regardless of the method by which the electron energy relaxation is measured due to inelastic collisions (including numerical method).

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Based on the results discussed above for the energy release area the following formula can be obtained

$$h_{ne}(E_0) = z_0 + \frac{\sqrt{3}}{\mathrm{K}\left(E_0\left[1 - \frac{z_0(E_0)\mathrm{K}(E_0)}{E_0}\right]\right)}. \quad (36)$$

3. Discussion of Results. Fig. 1 shows calculations of the mean electron beam energy with account of both types of deceleration $\overline{E}_{ne}(\bar{z}, E_0)$ and with account of only inelastic collisions $\overline{E}_n(\bar{z}, E_0)$ for initial energies of electron departures equal respectively to 10 and 5 keV. It is seen, that when elastic collisions are accounted for this leads to faster energy relaxation, which naturally shall result in higher peak values of energy release in a gas. Fig.2 referring to initial energy of 2 keV shows values $d\overline{E}_n(\bar{z}, E_0)/d\bar{z}$; $d\overline{E}_{ne}(\bar{z}, E_0)/d\bar{z}$ calculated by us and numerically determined by Monte Carlo method ($d\overline{E}_{ne}(\bar{z}, E_0)/d\bar{z}$) [9] and by method of function expansion by number of collision ($d\overline{E}_n(\bar{z}, E_0)/d\bar{z}$) [9]. The data presented evidences that analytical estimates satisfactorily match the results of numeric calculations [9]. In addition dependencies illustrated in Fig.2 prove that, firstly, value $d\overline{E}_{ne}(\bar{z}, E_0)/d\bar{z}$ is actually exceeding $d\overline{E}_n(\bar{z}, E_0)/d\bar{z}$. Secondly, this value is first noticeably increasing and then after reaching its maximum drops to zero.

Let's discuss physical reasons behind the indicated regularities. When only inelastic collisions are taken into account, the electron beam energy is relaxed upon electrons' normal movement towards the departure plane $z = 0$, i.e. in the volume element $dV = dxdydz$ located at point $A_1(x_1, y_1, z_1)$ energy is released only partially from the electrons which departed from plane element $dS = dydx$ from point $A_2(x_1, y_1, 0)$. The account of elastic collisions changing electron directions leads to the situation when electrons emitted from other points of the plane $z = 0$ with coordinates $x \neq x_1; y \neq y_1$ get into the volume element dV (Fig. 3). This naturally leads to increased energy release in the volume element dV. It is clear that the smaller is the average cosine of an angle between the normal to the plane of departure and the electrons' direction $\overline{\mu}_+(z_1)$, the greater is (at the fixed coordinate z_1) the departure region of the plane, "visible" from the point A_1. Thus reduction in $\overline{\mu}_+(z_1)$ leads to faster energetic relaxation of electrons. This is a reason why at the beginning $d\overline{E}_{ne}(\bar{z}, E_0)/d\bar{z}$ is increasing. After $\overline{\mu}_+(z_1)$ reaches its maximum value equal to 0.5, the increase in specific energy release halts.

In study [7] it was experimentally proven that dependency of the energy release zone sizes $R(E_0)$ and initial segment of the dependency $\overline{E}_1(\bar{z}, E_0)$ in the range of initial energies $5\ \text{keV} < E_0 < 54\ \text{keV}$ can be well illustrated by the following correlations:

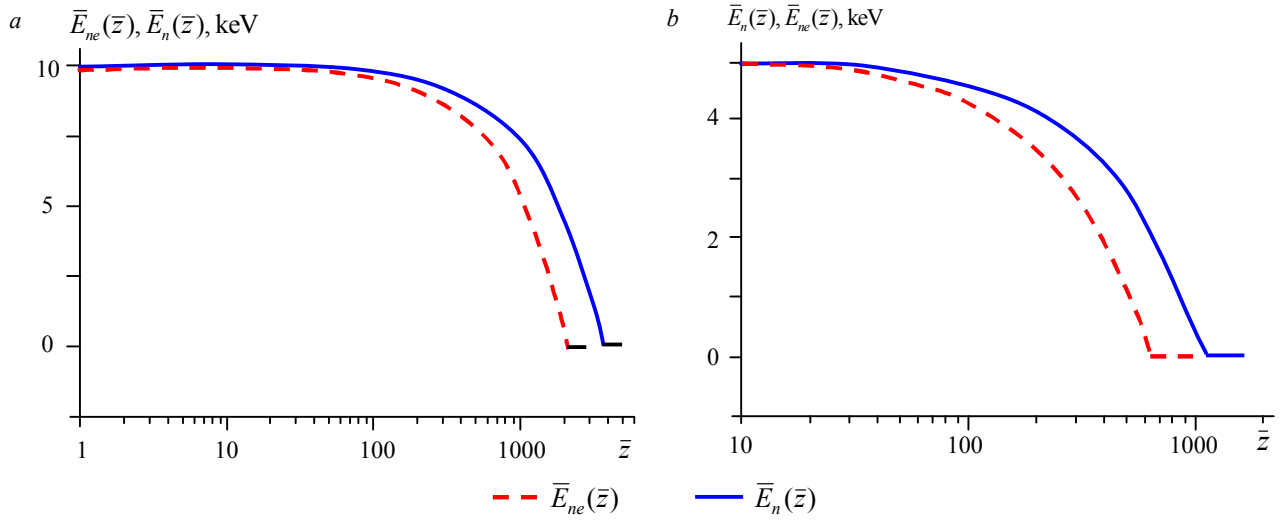


Fig.1. Dependency of the mean electron energy on the number of collisions calculated with account of both elastic and inelastic collisions $\overline{E}_{ne}(\bar{z})$ and of only inelastic collisions $\overline{E}_n(\bar{z})$ at initial energy of electrons equal to 10 keV (*a*) and 5 keV (*b*)

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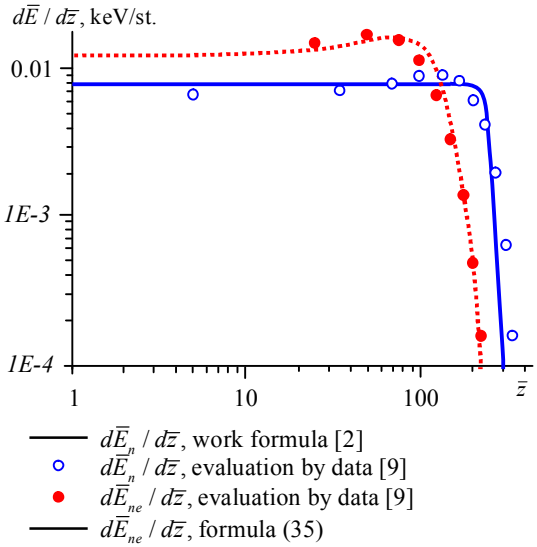


Fig.2. Dependency of specific energy release on the number of collisions (coll.), calculated with account of both elastic and inelastic collisions $d\bar{E}_{ne}/d\bar{z}$ and only inelastic collisions $d\bar{E}_n/d\bar{z}$ at initial energy of electrons equal to 2 keV

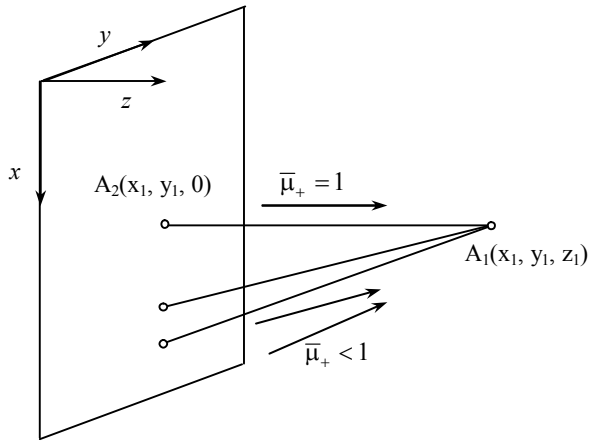


Fig.3. Explanation to correlation between $d\bar{E}_{ne}(\bar{z})/d\bar{z}$ and $d\bar{E}_n(\bar{z})/d\bar{z}$

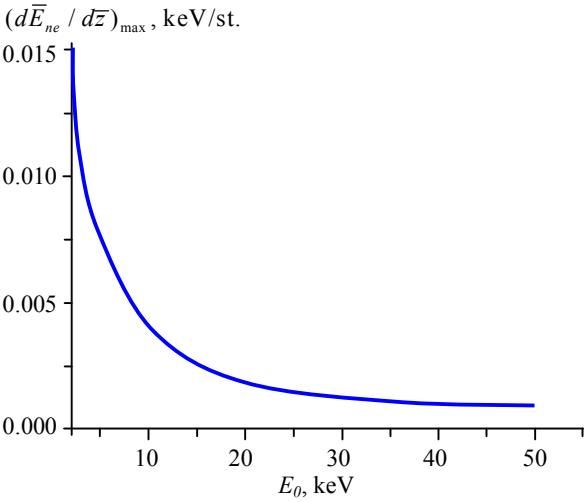


Fig.4. Dependency of maximum specific energy release on electron initial energy

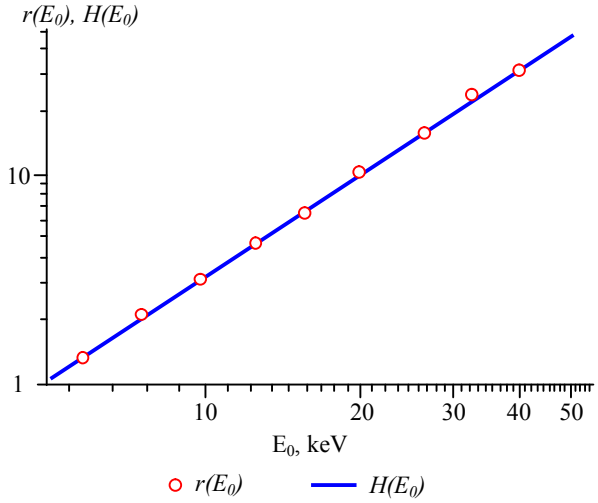


Fig.5. Comparison between values of energy release zone relative width $r(E_0)$ и $H(E_0)$ respectively measured by the author of paper [7] and calculated by us

$$R(E_0) = \alpha_1 E_0^{1.66}; \quad \bar{E}_1(\bar{z}, E_0) = E_0\left(1 - \frac{\bar{z}}{R(E_0)}\right)^{0.6}. \tag{37}$$

From the second correlation (37) the below correlation can be easily obtained

$$\frac{d\bar{E}_1(0, E_0)}{d\bar{z}} = \bar{E}_1'(\bar{z}, E_0)_{\bar{z}=0} = \alpha_2 E_0^{-0.66}, \tag{38}$$

where α_1, α_2 are certain numeric coefficients.

Fig.4 shows a comparative data for the maximum specific energy release in a gas at different initial energies of electrons, and Fig.5 and 6 show comparative data for calculated values not dependent on the constants α_1, α_2, respectively:

$$r(E_0) = R(E_0)/R(5keV); \quad H(E_0) = h_{ne}(E_0)/h_{ne}(5keV);$$

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$$e(E_0) = \overline{E}_{ne}{}'(\bar{z}, E_0)_{\bar{z}=0} / \overline{E}_{ne}{}'(\bar{z}, 5keV)_{\bar{z}=0};$$

$$D(E_0) = \overline{E}_1{}'(\bar{z}, E_0)_{\bar{z}=0} / \overline{E}_1{}'(\bar{z}, 5keV)_{\bar{z}=0}.$$

It is seen that analytical calculations presented in this paper appear to be good description for experimental results [7]. Furthermore, with increase in E_0 the value of $d\overline{E}_{ne}(\bar{z}, E_0)/d\bar{z}$ is decreasing, but zone size (calculated for energy of electrons E_0 and expressed in the number of inelastic collisions), where the major energy is released, is expanding. This is due to the fact that with increase in the departure energy the mean energy transferrable in electron collision with gas atoms is decreasing. Such decrease occurs non-lineary. In addition when electron departure energy is increased, the size of zone where electron distribution function undergoes isotropization is increasing. The described regularities lead to the observed dependency of $d\overline{E}_{ne}(\bar{z}, E_0)/d\bar{z}$ from E_0.

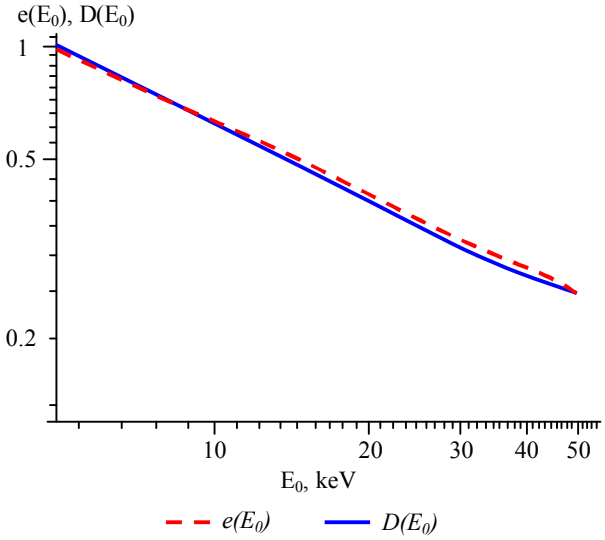


Fig.6. Comparison of specific electron energy release $e(E_0)$ and $D(E_0)$, respectively measured by the author of paper [7] and calculated by us

Conclusion. Below we summarize the main results of the study.

•A physical model of fast electron energy relaxation in a gas with account of elastic and inelastic collisions has been proposed. The calculated mass deceleration ability of the air is well illustrating the known experimental data.

• It has been revealed that it's important to take into account the elastic collisions to measure electron beam energy relaxation. In particular this leads to a significant increase in released energy and a decrease in the range where major energy release occurs.

• The analytical expressions have been obtained for mean flow energy and rate of energy release in a gas, which significantly (by dozens of times) reduce the time spent on measuring specific parameters of the energy release zone as compared to calculations using Monte Carlo method.

• The calculations have been made to assess energy release rates at different initial electron emission energies. Results are in a satisfactory accord with the known experimental data and numeric calculations by method of distribution function expansion by number of collisions.

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