21. Теоремы, относящіяся до уравненія Монжа и Ампера.

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§ 1. Дается дифференціальное уравненіе

(1) $Hr + 2Ks + Lt + M + N(rt - s^2) = 0,$

гдѣ

$$H,\ K,\ L,\ M,\ N.$$

суть функціи

$$x,\ y,\ z,\ p,\ q;$$

$$p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y};$$

$r,\ s,\ t$ имѣютъ общепринятыя значенія:

$$r = \frac{\partial p}{\partial x},\ s = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x},\ t = \frac{\partial q}{\partial y}.$$

Положимъ, что уравненіе (1) имѣетъ интегралъ

(2) $V(x,\ y,\ z,\ p,\ q) = 0.$

Тогда уравненіе (2) даетъ

$$(3)\quad \begin{aligned} &\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} + r\frac{\partial v}{\partial p} + s\frac{\partial v}{\partial q} = 0,\\ &\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} + s\frac{\partial v}{\partial p} + t\frac{\partial v}{\partial q} = 0. \end{aligned}$$

Называя для краткости

$$\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} = \Delta_x,\ \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} = \Delta_y,$$

изъ (3) имѣемъ

(4) $t\Delta_x - s\Delta_y + \frac{\partial v}{\partial p}(rt - s^2) = 0.$

Исключая

$$rt - s^2$$

изъ (1) и (4), получаемъ:

(5) $H\frac{\partial v}{\partial p}r + (2K\frac{\partial v}{\partial p} + N\Delta_y)s + (L\frac{\partial v}{\partial p} - N\Delta_x)\,t + M\frac{\partial v}{\partial p} = 0.$

Уравненія (3) и (5) линейны относительно

$$r,\ s,\ t.$$

Рѣшая уравненія (3) и (5) относительно $r,\ s,\ t,$ по правиламъ, излагаемымъ въ теоріи опредѣлителей, имѣемъ:

$$r = \begin{vmatrix} \frac{\partial v}{\partial q}, & 0, & \Delta_x,\\ \frac{\partial v}{\partial p}, & \frac{\partial v}{\partial q} & \Delta_y,\\ (2K\frac{\partial v}{\partial p} + N\Delta_y), & (L\frac{\partial v}{\partial p} - N\Delta_x), & M\frac{\partial v}{\partial p}. \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{\partial v}{\partial q}, & 0, & \Delta_x,\\ 0, & \frac{\partial v}{\partial q}, & \Delta_y,\\ H\frac{\partial v}{\partial p}, & (L\frac{\partial v}{\partial p} - N\Delta_x), & M\frac{\partial v}{\partial p}. \end{vmatrix} =$$

$$- \begin{vmatrix} \frac{\partial v}{\partial q}, & 0, & \Delta_x,\\ 0, & \frac{\partial v}{\partial q}, & \Delta_y,\\ H\frac{\partial v}{\partial p}, & (L\frac{\partial v}{\partial p} - N\Delta_x), & M\frac{\partial v}{\partial p}. \end{vmatrix} =$$

$$t = \begin{vmatrix} \frac{\partial v}{\partial p}, & \frac{\partial v}{\partial q}, & \Delta_x,\\ 0, & \frac{\partial v}{\partial p}, & \Delta_y\\ H\frac{\partial v}{\partial p}. & (2K\frac{\partial v}{\partial p} + N\Delta_y), & M\frac{\partial v}{\partial p} \end{vmatrix} =$$

[1]) Iordan, Cours d'Analyse, 1887, t. III, p. 364.

$$1$$

$$-\begin{vmatrix} \frac{\partial v}{\partial p}, & \frac{\partial v}{\partial q}, & 0, \\ 0, & \frac{\partial v}{\partial p}, & \frac{\partial v}{\partial q}, \\ H\frac{\partial v}{\partial p}, & (2K\frac{\partial v}{\partial p}, + N\Delta_y), & (L\frac{\partial v}{\partial p} - N\Delta_x) \end{vmatrix}.$$

Отсюда, замѣчая, что

$$rdx + sdy = dp \text{ и}$$
$$sdx + tdy = dq,$$

получаемъ

$$(6)\quad \begin{aligned} &\left[N\Delta^2{}_x + (2K\frac{\partial v}{\partial q} - L\frac{\partial v}{\partial p})\,\Delta_x + L\,\frac{\partial v}{\partial q}\,\Delta_y - M\left(\frac{\partial v}{\partial q}\right)^2\right]dx + \\ &\left[M\,\frac{\partial v}{\partial p}\,\frac{\partial v}{\partial q} + N\Delta_x\Delta_y - H\,\frac{\partial v}{\partial q}\,\Delta_x - L\,\frac{\partial v}{\partial p}\,\Delta_y\right]dy = \\ &= \left[L\left(\frac{\partial v}{\partial p}\right)^2 + H\left(\frac{\partial v}{\partial q}\right)^2 - 2K\,\frac{\partial v}{\partial p}\,\frac{\partial v}{\partial q} - N\,\frac{\partial v}{\partial p}\,\Delta_x - \right. \\ &\left. - N\,\frac{\partial v}{\partial q}\,\Delta_y\right]dp. \end{aligned}$$

$$(7)\quad \begin{aligned} &\left[M\,\frac{\partial v}{\partial p} + N\Delta_x\Delta_y - H\,\frac{\partial v}{\partial q}\,\Delta_x - L\,\frac{\partial v}{\partial p}\,\Lambda_y\right]dx + \left[N\Delta^2{}_y + \right. \\ &\left. + 2K\,\frac{\partial v}{\partial p}\,\Delta_y + H\,\frac{\partial v}{\partial p}\,\Delta_x - H\,\frac{\partial v}{\partial q}\,\Delta_y - M\left(\frac{\partial v}{\partial p}\right)^2\right]dy = \\ &\left[L\left(\frac{\partial v}{\partial p}\right)^2 + H\left(\frac{\partial v}{\partial q}\right)^2 - 2K\,\frac{\partial v}{\partial p}\,\frac{\partial v}{\partial q} - N\,\frac{\partial v}{\partial p}\,\Lambda_x - \right. \\ &\left. - N\,\frac{\partial v}{\partial q}\,\Delta_y\right]dq. \end{aligned}$$

Равенства (6) должны существовать при всѣхъ значеніяхъ

$$L,\ M,\ N,\ H,\ K.$$

Уравнивая въ (6) нулю коэффиціентъ при M, получаемъ:

$$(A)\qquad \frac{dx}{\frac{\partial v}{\partial p}} = \frac{dy}{\frac{\partial v}{\partial q}}.$$

Уравнивая въ (6) нулю коэффиціентъ при K, получаемъ:

$$(B)\qquad \frac{dx}{\frac{\partial v}{\partial p}} = \frac{-dp}{\Delta x}.$$

Уравнивая въ (7) нулю коэффиціентъ при L, получимъ:

$$\frac{dx}{\frac{\partial v}{\partial p}} = \frac{-dq}{\Delta y}.$$

Значитъ

$$(8)\qquad \frac{dx}{\frac{\partial v}{\partial p}} = \frac{dy}{\frac{\partial v}{\partial q}} = \frac{-dp}{\frac{\partial v}{\partial x} + p\,\frac{\partial v}{\partial z}} = \frac{-dq}{\frac{\partial v}{\partial y} + q\,\frac{\partial v}{\partial z}}$$

Принявъ во вниманіе (8), увидимъ, что (6) и (7) удовлетворяются тождественно. Выберемъ x и y такъ, чтобы каждый изъ четырехъ опредѣлителей третьяго порядка былъ равенъ нулю. Имѣемъ поэтому

$$(9)\quad \begin{aligned} &L\left(\frac{\partial v}{\partial p}\right)^2 + H\left(\frac{\partial v}{\partial q}\right)^2 - 2K\,\frac{\partial v}{\partial p}\,\frac{\partial v}{\partial q} - N\,\frac{\partial v}{\partial p}\,\Delta_x - \\ &- N\,\frac{\partial v}{\partial q}\,\Delta_y = 0, \\ &N\Delta^2{}_x + \left(2K\,\frac{\partial v}{\partial q} - L\,\frac{\partial v}{\partial p}\right)\Delta_x + L\,\frac{\partial v}{\partial q}\,\Delta_y - \\ &- M\left(\frac{\partial v}{\partial q}\right)^2 = 0, \\ &M\,\frac{\partial v}{\partial p}\,\frac{\partial v}{\partial q} + N\Delta_x\Delta_y + H\,\frac{\partial v}{\partial q}\,\Delta_x - L\,\frac{\partial v}{\partial p}\,\Delta_y = 0, \\ &N\Delta^2{}_y + 2K\,\frac{\partial v}{\partial p}\,\Delta_y + H\,\frac{\partial v}{\partial p}\,\Delta_x - H\,\frac{\partial v}{\partial q}\,\Delta_y - \\ &- M\left(\frac{\partial v}{\partial p}\right)^2 = 0. \end{aligned}$$

Принявъ во вниманіе (8), вмѣсто (9), получимъ

$$(10)\quad \begin{aligned} &\text{(а)}\ Ldx^2 + Hdy^2 - 2Kdxdy + Ndpdx + \\ &+ Ndqdy = 0, \\ &\text{(б)}\ Ndp^2 - 2Kdpdy + Ldpdx - Ldqdy - \\ &- Mdy^2 = 0 \\ &\text{(в)}\ Mdxdy + Ndpdq + Hdpdy + Ldqdx = 0, \\ &\text{(г)}\ Ndq^2 - 2Kdqdx - Hdpdx + Hdqdy + \\ &- Mdx^2 = 0. \end{aligned}$$

§ 3. Возьмемъ одно изъ уравненій (10) (безразлично какое) въ такой формѣ:

$$Mdxdy + N(rdx + sdy)(sdx + tdy) + Hdy(rdx + sdy + Ldx(sdx + tdy) = 0.$$

Получимъ

$$(Nrs + Ls)\,dx^2 + [M + N(rt + s^2) + Hr + Lt]\,dxdy + (Nst + Hs)\,dy^2 = 0.$$

Пользуясь (1) и сокращая на s, получимъ:

$$(Nr + L)\,dx^2 + 2(Ns - K)\,dxdy + (Nt + H)dy^2 = 0.$$

Отсюда

$$\frac{dx}{dy} = \frac{K - Ns \pm \sqrt{K^2 - 2KNs + N^2(s^2 - rt) - LNt - HNr - LH,}}{Nr + L}$$

или

$$\frac{dx}{dy} = \frac{K - Ns \pm \sqrt{K^2 + MN - LH,}}{Nr + L},$$

или

$$(12)\quad Ndp + Ldx - (K \pm \sqrt{K^2 + MN - LH}) \quad =$$

Рѣшая (11) относительно $\frac{dy}{dx}$, находимъ

$$\frac{dy}{dx} = \frac{K - Ns \pm \sqrt{K^2 + MN - LH}}{Nt + H}.$$

Отсюда

(13) $$Ndq + Hdy - (K \pm \sqrt{K^2 + MN - LH})\ dx = 0.$$

§ 4. Примѣръ I [1]).

$$(x+q)^2 + 2(x+q)(y+p)s + (y+p)^2 t + 2(x+q)(y+p) = 0.$$

Здѣсь

$$H = (x+q)^2,\ K = (x+q)(y+p),\ L = (y+p)^2,$$
$$M = 2(x+q)(y+p),\ N = 0,$$
$$K^2 + MN - LH = 0.$$

Уравненія (12) или (13) даютъ

$$\frac{dx}{x+q} = \frac{dy}{y+p}.$$

Уравн. (10) (в) даетъ

$$M(x+q)\,dy + H(y+p)\,dp + L(x+q)\,dq = 0,$$

или

$$2(x+q)\,dy + (x+q)\,dp + (y+p)\,dq = 0,$$

или

$$(x+q)\,dy + (y+p)\,dx + (x+q)\,dp + (y+p)dq = 0.$$

Интегрируя, получаемъ:

$$xy + qy + px + pq = \text{const}.$$

Это есть такъ называемый промежуточный интегралъ.

Это уравненіе даетъ систему:

$$\frac{dx}{x+q} = \frac{dy}{y+p} = \frac{-dp}{y+p} = \frac{-dq}{x+q}.$$

Отсюда

$$x + q = \text{const.},\ y + p = \text{const}.$$

Полагая

$$x + q = \alpha,\ y + p = \beta,$$

гдѣ α и β произвольныя постоянныя, имѣемъ:

$$p = \beta - y,\ q = \alpha - x;$$

$$dz = pdx + qdy = (\beta - y)dx + (\alpha - x)\,dy,$$
$$z = \beta x + \alpha y - xy + \text{const.},$$
$$z + (\alpha - x)(\beta - y) = \text{const}.$$

§ 4. Примѣръ II. Дается уравненіе

$$x^4 r - 4x^2 qs + 4q^2 t + 2px^3 = 0.$$

Здѣсь

$$H = x^4,\ K = -2x^2 q,\ L = 4q^2,$$
$$M = 2px^2,\ N = 0$$
$$K^2 + MN - LH = 0.$$

Уравненіе (12) даетъ

$$4q^2 dx + 2qx^2 dy = 0,$$

$$\frac{dx}{x^2} = \frac{-dy}{2q}.$$

Уравненіе (10) (в) даетъ

$$2pxdx + x^2 dp - 2qdq = 0,$$

или

$$pd(x^2) + x^2 dp - d(q^2) = 0,$$

$$px^2 - q^2 = \text{cons}\ t.$$

Отсюда

$$\frac{dx}{x^2} = \frac{dy}{-2q} = \frac{-dp}{2xp} = \frac{-dq}{0}.$$

Отсюда

$$q = \alpha,$$

гдѣ α произвольное постоянное.

Кромѣ того

$$\frac{dx}{x} + \frac{1}{2}\frac{dp}{p} = 0;$$

откуда

$$px^2 = \text{cons}\ t. = \beta.$$

Значитъ

$$dz = \frac{\beta}{x^2}dx + \alpha dy,$$

$$z = -\frac{\beta}{x} - \alpha y - \gamma,$$

гдѣ γ произвольная постоянная. Имѣемъ поэтому окончательно

$$\beta - zx - \alpha xy - \gamma x = 0.$$

[1]) Goursat, «Équations aux dérivès partielles du second ordre», 1896, p. 80.